THE EMERGENCE OF MODERN LOGIC

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> 'The end of logic is not the *syllogism* but *simple contemplation*. The *proposition* is, in fact, the means to this end, and the syllogism is the means to the proposition.' (Leibniz 1666, 75)

1. Introduction

Though in my presentation here I will try to be as objective as possible some of my biases will come through. Let me then lay some of my cards on the table.

Whereas most disciplines are first order disciplines formal logic is essentially not a first order discipline. What I mean by 'first order discipline' is that physics, chemistry, biology, geology, geography, psychology, economics, sociology and so on, explain a particular domain of reality; hence they are directly about the world. Though there are methodological considerations in these disciplines that are second order, they are essentially first order disciplines. Neither mathematics nor philosophy are first order disciplines in this way. That is not to say that mathematics and philosophy are not about the world. But they are about second order properties or concepts about the world. Logic in as much as it is concerned with the principles of inference is a first order discipline as these rules of inference are generally taken to be laws of thought and human minds are very much in the world. William Hamilton (1860, 3) puts it: 'Logic is the Science of Laws of Thought as Thought'. However, formal logic is a second order discipline as it is 'a science whose propositions are themselves second-order principles about principles of inference' (Kneale and Kneale 1962, 377). In formal logic then we do not just discover or formulate the principles of inference but also discover or formulate the principles that make these principles of inference the correct principles. This is crucial to the development of modern logic. In modern logic it is not sufficient to have primitives, definitions, rules of inference, axioms and theorems derived from the axioms; but we must also prove the consistency and completeness of the formal system; and even prove the deduction theorem which allows us to prove theorems to begin with. All of this is second order or even higher order activity.

Logic is also a purely formal discipline. Though the complete realisation and implementation of this did not happen until the nineteenth century, logic was always evolving towards this. An underrated figure in this regard was the German Moritz Wilhelm Drobisch (1802–1896) who called logic 'formal philosophy' and 'wrote that "logic is, in fact, nothing but pure formalism. It is not meant to be, and must not be, anything else."' (Vilkko, 207)

Logic is also not simply a tool for the sciences and other disciplines. Hence, I will perhaps marginalise people like Mill and empiricists and pragmatists who may give more emphasis to induction and abduction over deduction. These three features then of second order, formalism, and non-instrument are also features of mathematics and philosophy. It is no wonder then that the origins of logic are in mathematics (in the geometrical demonstrations of Thales and Pythagoras) and in philosophy (in the dialectic arguments of Parmenides, Zeno and Plato). And logic does not originate from the sciences as much as Aristotle may have wanted it to. Though mathematics and philosophy may be the parents of logic, logic has its own autonomy that makes it distinct from both her parents. George Boole, arguably the most important person in the development of modern logic puts this autonomy of logic clearly:

> I am then compelled to assert, that according to this view of the nature of Philosophy, *Logic forms no part of it*. On the principle of a true classification, we ought no longer to associate Logic and Metaphysics but Logic and Mathematics. […] Logic resting like Geometry upon axiomatic truths, and its theorems constructed upon that general doctrine of symbols, which constitutes the foundation of the recognised Analysis. […] Logic not only constructs a science, but also inquires into the origins and nature of its own principles,—a distinction which is denied to mathematics. (Boole 1847, 13)

The picture Boole presents here is somewhat different than what I have just given. He first divorces logic from philosophy and weds it to mathematics, and then he claims that logic is distinct from mathematics in that it inquires into the origins of the nature of its own principles. As Boole says he is referring to a notion of philosophy that was perhaps dominant at his time, however on an alternative interpretation of philosophy going back to Plato and Aristotle this feature of inquiry into the nature of its own principles is the core of philosophy; hence logic does indeed incorporate philosophy.

Now, we turn to the emergence of modern logic. There is often a debate as to whether modern logic emerged with George Boole (a mathematician) or Gottlob Frege (a philosopher). There is no doubt that without the golden age of mathematics of the eighteenth century modern logic would not have emerged. However, the role of philosophers in the development of modern logic cannot be ignored. Couturat claimed that it was neither Boole nor Frege who were the founders of modern logic, but it was another 'G' Gottfried Wilhelm Leibniz who was both a mathematician and a philosopher:

> En résumé, Leibniz a cu l'idée (plus ou moins précise, plus ou moins fugitive) de toutes les opérations de la Logique, non seulement de la multiplication, de l'addition et de la négation, mais mème de la soustraction et de la division. Il a connu les relations fondamentales des deux copules, à savoir:
>
> $(a < b) = (a = ab) = (ab' = 0)$.
>
> Il a trouvé la véritable traduction algébrique des *quatre* propositions classiques, et cela sous ses deux formes principales:

U.A.: Tout *a* est *b*	$a = ab$	$ab' = 0$.
U.N.: Nul *a* est *b*	$a = ab'$	$ab = 0$.
P.A.: Quelque *a* est *b*	$a \neq ab'$	$ab \neq 0$.
P.N.: Quelque *a* n'est pas *b*	$a \neq ab$	$ab' \neq 0$.

> Il a découvert les principales lois du Calcul logique, notamment les règles de composition et de décomposition. Enfin, il a très nettement conçu la double interprétation dont ce calcul est susceptible, suivant que les termes représentent des concepts ou des propositions et la parallélisme remarquable qui en résulte entre les propositions primaires et secondaires. Et un mot, il possédait presque tous les principes de la logique de Boole et de Schröder, et sur certains points il était plus avancé que Boole lui-meme. (Couturat, 1901, pp. 385–6)

> To sum up, Leibniz had conceived the idea (more or less precise, more or less radical) of all the operations of logic, not only of multiplication, addition and negation, but even of subtraction and division. He was acquainted with the fundamental relations of the two copulas known as:
>
> $(a < b) = (a = ab) = (ab' = 0)$.
>
> He found the correct algebraic translation of the four classical propositions, and these under their two principal forms:
>
> | | | |
> |---|---|---|
> | U.A.: All *a* is *b* | $a = ab$ | $ab' = 0$. |
> | U.N.: No *a* is *b* | $a = ab'$ | $ab = 0$. |
> | P.A.: Some *a* is *b* | $a \neq ab'$ | $ab \neq 0$. |
> | P.N.: Some *a* is not *b* | $a \neq ab$ | $ab' \neq 0$. |
>
> He discovered the principal laws of the logical calculus, notably the rules of composition and decomposition. Finally, he very clearly conceived that the double interpretation of this calculus is susceptible, next he found the terms representing concepts or propositions and the remarkable parallelism that resulted between primary and secondary propositions. In one word, he possessed almost all the principles of the logic of Boole and Schröder, and on certain points he was more advanced than Boole himself. (my translation)[1]

I am extremely sympathetic with Couturat's sentiments. Aristotle was a scientist and a philosopher but not a mathematician. That is why he could not invent modern logic though he did invent formal logic. Descartes was a mathematician and a philosopher but he was not a philosopher of the calibre of Plato and Aristotle as he did not realise that philosophy is essentially a second order discipline. Leibniz as a philosopher was of the highest calibre. He realised that not only was philosophy a second order discipline but it could be purely formal like mathematics. He therefore went in the search of a calculus, a logic that is, that would be the formal infrastructure for mathematics as well as for philosophy. This view today is called logicism.

The critiques of logicism object that if everything is reduced to logic, that is, if mathematics and philosophy are reduced to logic and pure formalism, then no ampliative knowledge could come about. However, Leibniz and later Frege actually claimed that analytic statements such as those of identities can be ampliative as are Leibniz's identity of indiscernibles and Frege's definition of number and of zero. Not only is ampliative knowledge possible in logic and mathematics, but such knowledge leads us to discoveries about the world. I think this connection from the inside of formal logic to the discovery of the outside world is fascinating and could be worked out though neither Leibniz nor Frege would have pushed this point. I will of course not be able to establish this here, not so much because of the paucity of time but because of my inability to do so at this moment. But twenty years down the road perhaps I will be able to establish this.

For now, let me just say that formalism, *albeit* pure formalism is the spinal chord of modern logic. So both Boole and Frege are the founders of modern logic. If a choice has to be made of course pure formalism and formal systems will be the domain of mathematics rather than of philosophy. Philosophers like VonWright and Hintikka nonetheless have made use of this formalism in the heart of philosophy in epistemology and ethics. Again I would claim that in their formalising of philosophy contemporary logicians are just as indebted to Plato, Leibniz and Kant as they are to Modern

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[1] I came upon the reference in Wolfgang Lenzen, 'Leibniz's Logic' (Gabbay and Woods, 2004, 7).

mathematics. Leibniz puts it best: 'Pure mathematics proves nothing against logic. For it has borrowed much from logic, and it also comes to the rescue of logic.' (1696, 470)

Modern logic may be characterised by the following essential characteristics:

a. variables, quantifiers, relations, functions;
b. axiomatisation;
c. truth functionality of logical connectives;
d. propositional calculus and predicate calculus—first order logic—set theory, second order logic;
e. formal semantics and metalogic: consistency, completeness and incompleteness;
f. development of modal logic, deontic logic and epistemic logic;
g. deviant logics: multivalued logic, fuzzy logic, quantum logic, paraconsistent logic.

I will now backtrack to the origins of logic in order to trace how these features emerged. I will be more concerned with (a) through (e) which can generally be characterised as the mathematisation of logic. It is also apparent to me that these five dimension are not explicitly combined to form formal symbolic logic until the mid or late Nineteenth century. So that is when modern logic emerges. We must hence examine why and how this emerged. It also seems apparent that that more than the contribution of philosophers, the emergence of modern logic was waiting for the active involvement of mathematicians. Kneale and Kneale (1962) make this point poignantly in their classic *The Development of Logic*:

> Although Leibniz had put forward a number of brilliant suggestions, it was not possible to make sure progress until mathematics had developed so far that the sort of abstraction he desired seemed natural and easy. When logic was revived in the middle of the nineteenth century, the new vigour came from mathematicians who were familiar with the progress of their own speciality, rather than from philosophers who were occupied with the controversies of idealism and empiricism. (p. 378)

So, as the golden age of mathematics was on in the eighteenth century mathematicians were too busy with developing the various branches of mathematics such as analysis, group theory, theory of functions and topology; and did not take time out to think about logic. When they did find time modern logic as mathematical logic emerged. Peckhaus describes this emergence as follows:

> Most nineteenth-century scholars would have agreed to the opinion that philosophers are responsible for research on logic. On the other hand, the history of late nineteenth-century logic clearly indicates a very dynamic development instigated not by philosophers but by mathematicians. A central outcome of this development was the emergence of what has been called the "new logic," "mathematical logic," "symbolic logic," or from 1904 on, "logistics." (Peckhaus 2004, 159)

What follows in the rest of my presentation is a merely unfolding of this paragraph.

I end the introduction with one note of caution on the word 'emergence'. I do not mean 'emergence' in any mystical sense or in a sense often used by some philosophers where at some point in time some phenomenon Y emerges out of X and is irreducible to X. I think the history of any discipline is much more democratic than it is often made out to be. When one properly studies the history one realises that it was not just George Boole (1815–1864) and Gottlob Frege (1848–1925) who invented modern logic but there were scores of mathematicians and philosophers such as Bernard Bolzano (1781–1848), William Hamilton (1788–1856), Charles Babbage (1791–1871), George Peacock (1791–1858), John Herschel (1792–1871), Augustus De Morgan (1806–1871), Alexander Bain (1818–1903), Duncan F. Gregory (1813–1844), Richard Dedekind (1831–1916), Charles Dodgson (1832–1898), William Stanley Jevons (1835–1882), Charles Saunders Peirce (1839–1914), Ernest Schröder (1841–1902), George Cantor (1845–1918), Giuseppi Peano (1858–1932, David Hilbert (1862–1943), and many others; who contributed to the invention of modern logic. Furthermore, the word 'emergence' is vague as we cannot pinpoint exactly when modern logic emerges. It surely did not emerge at the time of Leibniz or even Kant and it had surely emerged by the time of Russell and Whitehead's *Principia Mathematica*, the centenary of the publication of which is being celebrated this year. The emergence then took place somewhere in the fifty year span from the 1830s to the 1880s.

2. Origins of Formal Logic

Let us now briefly go back to the beginning. The word 'logic' is derived from the Greek '*logos*'. '*Logos*' means sentence or discourse. Broadly speaking logic then is the study of sentences or discourses. However, such a definition of logic would definitely be too broad as it covers grammar, semantics, pragmatics and linguistics. Though all of these may be related to logic they can be included in the philosophy of logic but not in logic proper. The more precise and generally accepted definition of 'logic' is that it is a science of inferences amongst sentences (or statements or judgments or propositions). It is the science of how certain sentences are derived from certain other sentences and what makes an inference valid.

2.1 *Origins of deduction*

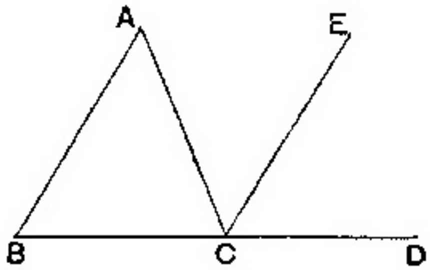
Among the Prescocratic philosophers we can distinguish three types of reasoning: demonstrations, dialectic and rhetoric.

Demonstrations can be seen in Thales (624–546 BCE) and Pythagoras (570–495 BCE) in their proofs of geometry and the birth of deduction. In a demonstration true conclusions are derived from true postulates with the aid only of primitive definitions and basic rules of inference. There is some controversy as to whether Thales and Pythagoras really came up with deductive proofs that could be called 'rigorous':

> In Thales the proofs of the theorems are either not given at all or are given without the rigor demanded in later times. […] Pythagoras […] concerning the right-angled triangle […] knew in the case of the triangle with sides 3, 4, and 5, without giving a rigorous proof. Euclid's is the earliest of the extant proofs of this theorem. […] The Pythagorians proved that the sum of the angles of a plain triangle is two aright angles. (Fink 1900, 194–5)

Rather than produce some fragments of proofs of Thales and Pythagoras whose authenticity may be questionable let us take the hint from Fink and look at a proof presented in the elements which surely must have come from earlier times. This is the proof of the famous proposition 32:

In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.



Let *ABC* be a triangle, and let one side of it *BC* be produced to *D*;

I say that the exterior angle *ACD* is equal to the two interior and opposite angles *CAB*, *ABC*, and the three interior angles of the triangle *ABC*, *BCA*, *CAB*, are equal to two right angles.

For let *CE* be drawn through the point *C* parallel to the straight line *AB*. [I.31]

Then, since *AB* is parallel to *CE*, and *AC* has fallen upon them, the alternate angle *BAC*, *ACE* are equal to one another. [I.29]

Again, since *AB* is parallel to *CE*, and the straight line *BD* has fallen upon them, the exterior angle *ECD* is equal to the interior and opposite angle *ABC*. [I.29]

But the angle *ACB* was also proved equal to the angle *BAC*;

therefore the whole angle *ACD* is equal to the two interior and opposite angles *BAC*, *ABC*.

Let the angle *ACB* be added to each;

therefore the angles *ACD*, *ACB* are equal to the three angles *ABC*, *BCA*, *CAB*.

But the angles *ACD*, *ACB* are equal to two right angles. [I.13]

therefore the angles *ABC*, *BCA*, *CAB* are also equal to two right angles.

Therefore etc.

Q.E.D.

(Euclid, 316–7)

After the mathematical demonstrations came Zeno's (490–420 BCE) famous dialectical arguments in the form of *reductio ad impossible*. These were also deductive. They differed from demonstrations in that whereas in demonstrations we begin with the premises and derive the conclusion deductively; in dialectic arguments we begin by assuming that a premise is true and derive a conclusion from them. More specifically in dialectic arguments as crafted by the Eleatic Parmenides (520–450 BCE), we first assume that *p* is true and derive a contradiction that follows from it, then we assume that ~*p* is true and derive a contradiction from it.

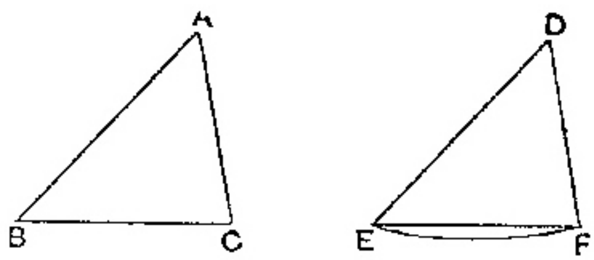
Here is an example of one of Zeno's paradoxes using a *reductio ad impossible* argument reconstructed by Aristotle. In which the assumption to be refuted is that (0) what is moving is at least at some moments not at rest. The argument then derives a contradiction:

(1) Anything occupying a place just its own size is at rest.
(2) In the present, what is moving occupies a place just its own size.
So, (3) in the present, what is moving is at rest.
Now (4) what is moving always moves in the present.
So (5) what is moving is always—throughout its movement—at rest. (Kirk et al. 1984, 273)

(0) and (5) form an explicit contradiction.

In Zeno's actual arguments as in the above one usually one side is done. He may assume (*p*) that an object moves from point s_1 *to* s_2 and derive a contradiction from it, so that *p* is not true. He usually does not assume (~*p*) that an object does not move from one point s_1 *to* s_2 and derive a contradiction from that. He need not do that once it is observed that an object moves from point s_1 *to* s_2 it is clearly false to say that the object does not mover from s_1 *to* s_2. This is why this is called a Zeno's paradox. An object o_1 obviously moves from point s_1 *to* s_2, but if you assume that it does then it leads to a contradiction. When the contradiction derived from an assumption p is an explicit contradiction of the form $q \wedge \sim q$ then $\sim p$ is proven to be true by *reductio ad impossible*. Later, in Plato we see *reductio ad absurdum* arguments in which from an assumption p any false proposition is derived (which is not necessarily a contradiction), so $\sim p$ is proved to be true. *Reductio ad impossible* arguments are a proper subclass of *reductio ad absurdum* arguments. Both are deductive, but some mathematicians have often been suspicious of *reductio ad absurdum* arguments that are not *reductio ad impossible*. However, if deduction in mathematics has the characteristics of demonstrations discussed above then all *reductio ad absurdum* arguments must be accepted as valid deductive arguments. That is because once a false proposition is derived in a demonstration then the starting point was not a true proposition. But then it was not a demonstration. Hence, mathematicians should not accept any *reductio* arguments of any time. Yet, *reductio* arguments are used even in Euclid's *Elements*:

> Proposition I.4 states:
> *If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lies equal, they all also have the base equal to the base, the triangles will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.*
> Proof:



In the middle of the proof we have:

> hence the base *BC* will coincide with the base *EF*.
> [For if, when *B* coincides with *E* and *C* with *F*, the base *BC* does not coincide with the base *EF*, two straight lines will enclose a space: which is impossible.
> Therefore the base *BC* will coincide with *EF*] and will be equal to it. [c.n. 4] (Ibid., 248)

Technically, there does not seem to be an explicit contradiction here, but the explicit contradiction can be brought out with a hidden assumption. Euclid's definition 4 states: 'A straight line is a line which lies evenly with the points on itself' (Ibid., 153). Hence, if *B* coincides with *E* and *C* coincides with *F*, then *BC* and *EF* are the same line and cannot enclose a space. So, if we label 'two straight lines will enclose a space' as q, then $\sim q$ is 'two straight lines do not enclose a space' and we can get the desired contradiction $q \wedge \sim q$.

Perhaps we have to give up the claim that all mathematical arguments are pure demonstrations. In any case in the Presocratics we find two types of deductive reasoning:

demonstrations which are solely mathematical and dialectical arguments which seem to be the nerve of philosophy. The Sophists raised scepticism against dialectical arguments perhaps because they seemed to lack the certainty of demonstrations. Instead, they promoted rhetoric as the art of convincing others of the truth of your claims through means that we may call 'fallacious reasoning'. Rhetoric may or may not be a form of reasoning, but it is definitely not deductive. Hence, from early on we can restrict the use of 'logic' as that pertaining only to deductive inferences hence rhetoric does not have a logic, whereas mathematics as demonstrations and philosophy as dialectical arguments do have a logic as they are both employing purely deductive reasoning. In order to avoid another type of scepticism I will define dialectic arguments as simply those arguments in which a conclusion is derived by assuming that the premises are true. To use a distinction we often make in the first few classes of elementary logic, correct philosophical dialectic arguments must be valid whereas correct mathematical demonstrations must be sound.

Aristotle of course was well aware of these two types of deduction and the logic that he formalised, namely syllogistic logic was common to both. Aristotle had an encyclopaedic mind and wore many hats. But at heart he was a scientist and being a scientist he saw more value in demonstrations which were mathematical than in dialectic arguments that were metaphysical. This preference for mathematics is also the reason for why he formally developed what would be close to modern predicate calculus and set theory without first developing a propositional calculus. Also being a scientist he was more concerned with general propositions than with particular propositions. Hence, his valid syllogisms were all given in terms of general propositions. Though his intentions were noble, the failure to develop a propositional calculus at that time impeded the progress of logic for centuries to come. But Aristotle is hardly to blame for this failure as he was not omniscient and he did not know what he needed to know in order to develop a propositional calculus as we shall see later.

Let me at this juncture let the cat out of the bag. It seems that the axiomatisation of arithmetic was necessary for the axiomatisation of propositional calculus, which in turn was necessary for the axiomatisation of predicate calculus; hence even though Aristotle comes close to the axiomatisation of predicate calculus he fails to accomplish the task. I can probably end my lecture at this point as I have pinpointed the reason for why modern logic emerged when it did. However, I will argue that as important as this reason may be, it is not the only reason, and further suggest that perhaps there are other important features of modern logic such as proofs of consistency and completeness which are equally essential and which are not an automatic consequence of the axiomatisation of logic. I could argue further, though I will not do that here, that even without the axiomatisation of propositional and predicate calculus, a logic with all the other features could be developed as a natural deduction system; though it would be extremely inconvenient and cumbersome.

2.2 *Propositions*

Though I will avoid a discussion of philosophy of logic, something must be said about what inferences hold between. Do they hold between sentences, or statements or propositions? In modern logic an inference may be expressed as follows:

Theorem 7.7 (Completeness) For every $\varphi \ni \mathsf{L}_K$

$\models \varphi$ implies $\vdash \varphi$ (van Ditmarsch et al., 181).

What do the symbols 'φ' and 'L_K' stand for? To answer this question we must enter a study of language. I quote none other than George Boole from *The Mathematical Analysis of Logic*:

> That which renders Logic possible, is the existence in our minds of general notions,—our ability to conceive of a class, and to designate its individual members by a common name. The theory of Logic is thus intimately connected with that of Language. A successful attempt to express logical propositions by symbols, the laws of whose combinations should be founded upon the laws of the mental processes which they represent, would, so far, be a step toward a philosophical language. But this is a view which we need not here follow into detail. (Boole 1847, 4–5)

One of the aspects that was not clear to Aristotle was the distinction between a sentence and a proposition. The Megarians who were contemporaries of Aristotle did have a clearer theory of meaning in which sentences referred to propositions. Aristotle did however a clearer conception of the truth values 'true' and 'false' as he states: 'To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true;' (*Metaphysics* 1011b 26–28); though he was not clear about whether the truth bearers were sentences, statements or propositions; but it surely was not facts as seen by the quotation. Facts are rather the truth makers. The Stoics following the Megarians sharpened the theory of meaning and came very close to claiming that propositions were thoughts which were the references of sentences and it was propositions which were true and false and not sentences. However, they could not come up with Frege's insight that 'true' and 'false' were referents of sentences. A theory of meaning however may only be a minor part of the development of propositional calculus.

2.3 *Use of variables in syllogistic logic*

Aristotle's syllogism first appears in Chapter 4, Book I of the *Prior Analytics*: 'If *A* is predicated of every *B*, and *B* of every *C*, *A* must be predicated of every *C*.' (26^b 1–2) This is the famous 'Barbara', that is, valid syllogism in the AAA in the first figure and usually depicted as:

All *M* are *P*.
All *S* are *M*.
Therefore, All *S* are *P*.

In using variables, which he did not use in earlier works like *De Interpretatione*, Aristotle realised the first essential feature of modern logic. However, the sharpness in the use of letters for variables was brought in by modern mathematicians. Here is an example:

> I then laid down the rules for the selection of letters. […]
> 1. All upright letters, as a, c, d, e, A, B, represent framing.
> 2. All inclined letters, as *a*, *c*, *d*, *e*, *A*, *B*, represent moveable parts.
> 3. All small letters represent working points. (Babbage, 143)

In Aristotle's logic we need to distinguish between capital letters when used for terms and when used for the categorical statements. So to honour this great man Charles Babbage without whose contribution I surely would not be typing this on the computer, I will use capital letters in italics as variables for terms and I will use capital letters 'A', 'E', 'I' and 'O' in normal font as names of the four types of categorical statements.

We must also note that in Aristotle's original formulation the syllogism would be better interpreted set theoretically, but of course this was not available to Aristotle. Aristotle realised that in order to decide which forms of syllogisms were valid all the possible forms will have to be tested (which are 256 in the revised Aristotelian system). This task would become cumbersome, tedious and practically close to impossible if variables were not used. More importantly, he realised that variables were necessary to capture the universality of syllogism. Consider the following syllogism:

All primates are mammals.
All humans are mammals.
Therefore, All humans are primates.

This is the syllogism in the mood AAA in figure 2. I hope that we can all see that this is invalid. Yet, in the example of the syllogism given here the premises as well as the conclusion are true, but the conclusion does not follow from the premises. How do we demonstrate the invalidity? This can be done by refutation by logical analogy:

All primates are mammals.
All cows are mammals.
Therefore, All cows are primates.

So, we have a counterexample as the premises are true and the conclusion is false. In the minor premise I have changed the subject term from 'humans' to 'cows'. In order to make this substitution I must first abstract from 'humans' to '*S*' where *S* is any term, and then I substitute for '*S*' cows. Furthermore, the form of each mood in each figure is more clearly understood when stated with variables:

All *P* is *M*.
All *S* is *M*.
Therefore, All *S* is *M*.

In fact when I wrote down the example and counterexamples I used this formula with variables of the syllogism in the second figure in the mood *AAA* in order to make my concrete construction. This shows the primacy of the abstract over the concrete in logic. It may seem that I am labouring a rather trivial point here, but in a way the move towards greater and greater formalisation is a move towards higher levels of abstraction. This is the point made by Kneale and Kneale throughout Chapter VI entitled 'Mathematical Abstraction' which along with Chapter VII presents the core of the emergence of modern logic:

> When one proposition follows logically from another, it should be possible to formulate the two propositions in such a way that their relation can be seen to depend on their form alone, that is to say, on their logical structure as opposed to their special subject-matter. (Kneale and Kneale 1962, 384)

The point I am labouring here is that this 'form alone' is seen more clearly when we use variables instead of concrete terms. It will of course be possible to do Aristotelian formal logic without the use of variables but it will be highly inconvenient. However, when we get to modern logic it will become impossible to do symbolic logic without variables, quantifiers and functions. Hence, there is a quantum leap from the convenience of variables in Aristotelian syllogistic logic to the necessity of variables in modern logic.

The defenders of Aristotle might ague that even in Aristotelian logic, the use of variables was indispensable. However, this point cannot be established convincingly, especially due to the hindsight we have now. Arithmetic was around at the time of Aristotle and all through the reign of classical logic. During this period algebra also emerged, but it took reflections on the formal character of algebra by mathematicians like George Peacock (1830) to set the ground for the algebrisation of logic, rather than the arithmetisation of logic, which was the key to the emergence of modern logic:

> […] the fundamental operations of Algebra are altogether symbolical, and we might proceed to deduce symbolical results and equivalent forms by means of them without any regard to the principles of any other science; and it would merely require the introduction of some such sign as = in the place of the words *algebraic result of*, or, *algebraically equivalent to*, to connect the results obtained with the symbolic representation of the operations which produce them, in order to supersede altogether the use of ordinary language. (Peacock 1830, xi)

Algebra then is closer to the purely formal discipline that modern logic is likely the offspring of than arithmetic is. We begin then to find a dent in the supposed intimate connection between arithmetic and logic.

2.4 ***Limitations of syllogistic logic***

Aristotle considers the four moods in the first figure *AAA*, *AII*, *EAE* and *EIO* to be the basic forms of valid syllogisms and other valid syllogisms may be reducible to these forms. There were further attempts in Aristotelian logic to reduce all syllogisms to *AAA* in the first figure. This clearly indicates that Aristotle had a notion of axiomatics. Furthermore, his repeated emphases on first principles in other writings also points to axioms. Aristotle was also quite aware that geometry could be axiomatised even though Euclid's *Elements* had not yet appeared. Even though Aristotle could clearly see the possibility of axiomatisation of syllogistic logic, he failed to present his logic in an axiomatic system. Even after the availability of Euclid's *Elements* for centuries classical logic failed to be axiomatised. This failure was due to the limitations of Aristotelian logic. Even though Aristotle's three laws of logic are universal laws of thought, his syllogistic logic falls short of global application.

The first limitation is the restriction of the form of all syllogisms to two premises, major and minor and the conclusion. Let us take a look at the proof of Euclid's famous proposition 32 above. How can this proof be reduced to syllogistic logic? Even if this were possible it would be a practical nightmare to achieve it. One way would be to go part by part of the proof and put it in syllogistic form. This would end up with this as the final syllogism where the major premise and the minor premise themselves would already have been established through valid syllogisms:

> All triangles in which line *BC* can be produced to *D* are triangles of the type *ABC* .
> All triangles in which the exterior angle *ACD* is equal to the two interior and opposite angles *CAB*, *ABC*, and the three interior angles of the triangle *ABC*, *BCA*, *CAB*, are equal to two right angles are triangles in which *BC* can be produced to *D*.
> Therefore, All triangles of type *ABC* are triangles in which angle *ACD* is equal to the two interior and opposite angles *CAB*, *ABC*, and the three interior angles of the triangle *ABC*, *BCA*, *CAB*, are equal to two right angles.

The second limitation is that only subject–predicate propositions can be used in syllogistic logic. In reducing it is difficult to represent part of the proof into a syllogism when the sub proof involves a relation. Furthermore, the recursive complete proof of proposition 32 requires all five of Euclid's axioms. How are the axioms to be captured syllogistically? The task becomes difficult because the first three postulates of Euclid are constructions and not propositions:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles of the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles. (Euclid, pp. 154–5)

Here is an attempt to syllogise Euclid's first postulate:

All pairs of points are points between which a line q_1q_2 can be constructed.
All points (q_1, q_2) are pairs of points.
Therefore, All points (q_1, q_2) are pairs of points are points between which a line q_1q_2 can be constructed.

Even though we have represented the first postulate as *AAA* in the first figure, the formulation is questionable because the middle term seems to be identical to the subject term. Though this is not technically incorrect usually in Aristotelian logic all three of the major term, minor term and middle terms must be distinct. However in Saccheri's profound *reductio* argument against the validity of AEE in the first figure we find a similar identification of the subject term with the middle term:

Every syllogism with a universal major and an affirmative minor premises yields a conclusion in the first figure.
But no syllogism of the pattern *AEE* has a universal major and an affirmative minor premiss.
Therefore, No syllogism of the pattern *AEE* yields a conclusion in the first figure. (Kneale and Kneale 1962, 346)

The identity of the subject term and middle term here may not be explicit, but by definition *AEE* does not have a universal major and affirmative minor premise, rather it is composed of a universal affirmative major and a universal negative premise. Hence, it immediately reduces to an identity. It is interesting to note that this *reductio* does not look like Zeno's *reductio* arguments where we begin with the negation of what we want to prove as the assumption. However, this argument itself is in the mood *AEE* in the first figure. So, the *reductio* does indeed begin with the assumption that *AEE* in the first figure is a valid argument form. However, if it is valid then as the conclusion here states it is not valid. Hence, we have the explicit contradiction that a *reductio ad impossible* argument requires.

A third limitation as mentioned earlier is that syllogistic logic is not built on a propositional calculus and proofs like that of Proposition 32 when represented purely logically employ propositional calculus. Aristotle was inspired by mathematical demonstrations rather than by dialectical arguments in developing syllogistic logic as he clearly states when he first introduces syllogisms: 'After these distinctions we now state by what means, when, and how every deduction is produced; subsequently we must speak of *demonstration*. (my emphasis) (*Prior Analytics*, 25^b 26–7). It is hence ironic that he failed to capture geometrical demonstrations in syllogistic logic. Despite being an expert in almost every discipline Aristotle was not a mathematician and this also explains his failure to axiomatise logic. However, if we are to accept this as the reason, then we may wonder why in the next two thousand years mathematicians like Descartes, Pascal and Leibniz could not axiomatise logic. The better explanation is that despite this long standing axiomatisation of geometry it took the developments of mathematics in the eighteenth and nineteenth centuries for the axiomatisation of arithmetic to emerge. And the axiomatisation of arithmetic and set theory coincides with the axiomatisation of logic. This supports the logicist thesis that arithmetic is reducible to logic. Again, I could stop my lecture here as I have given the essential reason why modern logic emerged when it did. But I request you to be patient because after repeating this point a few times I may end up challenging the legitimacy of it.

2.5 *Failure of classical logic to capture truth functionality of logical connectives*

The Megarians explicitly realised the truth functionality of compound statements involving the connectives conjunction, disjunction and conditional as Philo (4^{th} century BCE) states: ‘a sound conditional is one that does not begin with a truth and end with a falsehood, e.g. when it is a day and I am conversing the statement “if it is day I am conversing”. (Kneale and Kneale, 128) I have purposefully picked the conditional as an example because all throughout the history of logic the conditional has somehow been tied with implication and inference, so that the truth functional representation of what in modern logic is called the ‘material conditional’ is rightly motivated as the only case that is definitely false is when the antecedent is true and the consequent is false which is related to the conclusion of an argument must be true if the premises are true. However, from the Megarians to the Stoics through medieval philosophy all the way up to and into Modern philosophy the truth functionality of logical connectives could not be formalised. It could not be formalised until logic was algebrised and this algebrisation of logic depended on the development of algebra itself to the level of an abstraction in which it was no longer restricted to quantity alone. This development was made by George Peacock (1830) and Duncan Gregory (1840), which further led to Boole’s development of algebra. Simply put it was the algebra of operations and particularly of the binary operations of multiplication and addition that could be used to represent the logical connectives of conjunction and disjunction. Boole provided all the possibilities on which truth tables for logical connectives could be built and Frege rode piggy back on this to actually lay down the truth tables; hence grounding truth functionality formally as required for the development of modern logic.

.6 *Failure of classical logic to develop a propositional and predicate calculus, set theory and second order logic*

The above sections make it clear that the lack of a clear distinction between sentences and propositions, the limitation of subject–predicate propositions to capture relational propositions, a lack of a formal development of truth functional connectives were the factors because of which propositional calculus could not be developed in Aristotelian logic. Even though Aristotle’s syllogistic logic is predicate logic it does not provide a predicate calculus. First, a predicate calculus presupposes a sentential calculus which was not available to classical logic. Second, the use of quantifier symbols (∃x) and (∀x) was not available until after Hamilton and De Morgan developed earlier notations and Frege developed the notation that we use today. How can a lack of notation account for the failure of the emergence of predicate calculus? Without these symbolisations certain implications that hold in Aristotelian logic were simply mistaken. An A proposition implies and I proposition in classical logic. So, ‘all unicorns are one-horned’ implies that ‘some unicorns are one-horned’. However, ‘all unicorns hare one-horned’ is true whereas ‘some unicorns are one-horned’ is false because there are no unicorns. To avoid this Aristotle was led to say that ‘all unicorns are one-horned’ is false because ‘all *A* are *B*’ cannot be true unless there exists at least one *A*. If the quantifier notations were available, perhaps Aristotle would have realised his error. Similarly, the proposition ‘no humans are insects’ is true now but it would not have been true three million years ago since no humans existed, whereas in modern logic this statement would always be true whether or not humans existed.

Classical logic could also have been developed as a logic of classes but as we have seen above Aristotle and his followers for two millennia did not have the mechanism to develop set theory. Even though the notion of classes, membership, inclusion, union and intersection were there, the symbols ‘$\{x_1, x_2, \ldots\}$’, ‘∈’, ‘⊂’, ‘∪’, ‘∩’ were missing. Additionally, the lack of axioms made it impossible for set theory to be developed. Even if propositional logic, predicate calculus and set theory could have been developed within Aristotelian logic there would have been no realisation that all of this was first order logic as there is not indication that there was any realisation that there could be a second order and higher orders of formal logic as this is solely a realisation of modern logic.

.7 ***Failure of classical logic to develop a formal modal logic***

Aristotle and classical logic since Aristotle did develop modal logic to some extent. Aristotle clearly specified modal syllogisms. Three were however three types of confusions. First, the confusion of what the meaning of ‘necessary’ was. Second, the confusion of whether the ‘necessary’ and ‘possible’ operated on propositions or on predicates. Even if this confusion can be cleared up what Aristotle and logicians all the way up to the twentieth century did not have available to them were the symbols ‘’ and ‘◇’ for necessary and possible respectively. Again, one may wonder how an appropriate notation can lead to the development of logic. But this is exactly what has happened. In system S4 of modal logic the additional axiom to system T is *p* → *p*. *p* is really (*p*). that means that the first is the necessity operator on the proposition which itself is a proposition with a necessary operator. Without the notation this literal translation of ‘*p*’ as ‘necessary necessary *p*’ which is a proposition is obscure or meaningless. It can better be stated as ‘necessary *p* is necessarily true’. Somehow this still does not capture the meaning that '$\Box\Box p$' does. Now, if one is suspicious of the truth of the axiom one need only substitute the first '$\Box$' by its definition '$\sim\Diamond\sim$'. The axiom now becomes '$\Box p \rightarrow \sim\Diamond\sim\Box p$' which perhaps now seems obvious. In English this translates as 'if *p* is necessarily true then it is not possible that *p* is not necessarily true' whereas the original axiom reads as: 'if *p* is necessarily true then it is necessary that *p* is necessarily true.' Even though the second form of the axiom does not become more obvious than the first form until we translate it into English, my point is that using language alone and without the notations for modal operators this axiom could not have been realised. Whether or not one agrees with me here it seems that history is witness to my assessment as this axiom was not realised until the twentieth century.

The development of modal logic is antecedent to the development of deontic and epistemic logics. Since ethics and epistemology have been at the core of philosophy ever since Plato, surely philosophers would have lived to have developed deontic and epistemic logics, but since they did not have a developed modal logic they could not have had a developed deontic and epistemic logic.

.8 *Failure of classical logic to develop a metalogic*

Aristotle's reflective discussions on the syllogism and his determination of the valid forms as the only valid forms were traces of proving consistency and completeness. However, there is not even an inkling of the formal development of this in Aristotle and hardly any progress is made towards this until the end of the nineteenth century and it is completely realised only by the Hilbert program.

The liar's paradox is indeed the generator of Gödel's incompleteness theorem. The liar's paradox is first attributed to Eubulides (4^{th} Century BCE), a Megarian who was a contemporary of Aristotle. Even though there were many attempted solutions to this paradox no solution was satisfactory and no one ever had the insight that the incompleteness of a formal system could be demonstrated with the formal system. Why could this not be done? Gödel's incompleteness requires three steps: the first is to formally prove the soundness, consistency and completeness of the formal system. Soundness means that every theorem of L is a tautology, consistency is established when it is proven that if α is a thesis of L then $\sim\alpha$ is not a thesis and vice versa, and completeness is when every tautology of L is a theorem of L. The second step is to formulate the incompleteness theorem with the formal system, which basically says that there is a tautology in L that is not a theorem of L, that it cannot be proven in L. The third step is the proof which establishes that if L is consistent then it is incomplete. None of the three steps were available in Aristotelian logic. This is also the reason why within the limitation of Aristotelian logic the liar's paradox could not be solved. It could not be solved because it could not be formally represented.

.9 *Failure of classical logic to develop deviant logics*

Again, there are probably traces of multivalued logics, fuzzy logic and paraconsistent logic (in Heraclitus and Hegel) but again not even a grain of a formalised account of these logics had been developed.

3. The Roots of Modern Logic in Leibniz

What follows in sections 3.1 through 3.5 is an unfolding of Couturat's quotation from section 1 above where he claimed that Leibniz had somewhat developed all the essential features of modern logic and in some ways he went beyond Boole.

3.1 The universal calculus

Leibniz was well aware of the limitations of classical logic. He hence began the construction of a universal calculus that would be wider and more comprehensive basis for logic than what we have in classical logic. Leibniz had in mind a formal calculus that would be the basis not only of mathematics but of philosophy as well:

> If this is done, whenever controversies arise, there will be no more need for arguing among two philosophers than among two mathematicians. For it will suffice to take the pens into the hand and to sit down by the abacus, saying to each other: *Let us calculate*. (Leibniz, translated by Lenzen 2004, 1)

The universality of this calculus is stressed repeatedly by Leibniz:

> Once our men carry the method through to the end, therefore, they will always philosophise in the manner of Boyle, except in so far as nature itself, to the degree to which it is known and can be subjected to the calculus and to the degree that new qualities are discovered and reduced to this mechanism, will also give to geometricians new material to which to apply it. (1675, 166)

> I have suggested elsewhere that there is a calculus more important than those of arithmetic and geometry which depends on the analysis of ideas. This would be a universal characteristic, and its formation seems to me one of the most important things that can be undertaken. (1702, 585)

> […] I should still hope to create a *universal symbolistic* [*spécieuse générale*] in which all truths of reason would be reduced to a kind of calculus. At the same time this could be a kind of universal language or writing, though infinitely different from all such languages that have thus far been proposed, for the characters and the words themselves would give direction to reason, and the errors—except those of fact—would be only mistakes in calculation (1714–15, 654)

Had Leibniz anticipated the computational theory of mind?

We began with the distinction between two types of deduction: demonstrations and dialectical arguments. Whereas Aristotle wanted to consider both on par as both involved syllogistic reasoning, even though he gave preference to mathematical demonstrations; Leibniz on the other hand claims that: 'Perfect demonstrations are possible in all disciplines' (1666, 74) Hence, all pure reasoning can be reduced to demonstrations and that is the guiding light for his universal calculus.

3.2 Logic is purely formal

For Leibniz logic was a purely formal discipline. Furthermore, the rigour of all the sciences is derived from the formalism that is incorporated into it. So, even though each science has its autonomy according to its domain, the rigour of it is derived from the formal system that the discipline imports. Leibniz's inspiration for logic being purely formal was surely not from Descartes but from Hobbes:

> […] if reasoning chance to be nothing more than the uniting and stringing together of names or designations by the word 'is'? It will be a consequence of this that reason gives us no conclusion about the nature of things, but only about the terms that designate them, […] according to which we join these names together. (from Kneale and Kneale 1962, 311)

The purely formal character of logic outlined by Hobbes is that logic is not about terms and not about the objects designated by the terms and that it is a purely formal discipline about the combinations of terms which are formally defined. Inspired by Hobbes Leibniz also thought that symbols could be introduced through definitions, though he allowed for real as well as nominal definitions whereas for Hobbes all definitions were nominal and arbitrary. And once the symbols are introduced they could be subjected to combinatory analysis, and such combinatory analysis was the basis of Leibniz's universal calculus of logic:

> Algebra […] is only a part of this general device […] {in} algebra truth can be grasped as if pictured on paper with the aid of a machine. […] {whatever} algebra proves is due to a higher science […] *combinatorial characteristic* […] nothing more effective can well be conceived for perfecting the human mind and if this basis for philosophising is accepted, there will come a time, […] when we shall have certain knowledge of God and the mind as we now have of figures and numbers and when the invention of machines will be no more difficult than the construction of geometric problems. (1675, 166)

Here, Leibniz does not only anticipate the coming of the algebra of logic but also how it will come about. The first step, as we will soon see, in the emergence of the algebra of logic is a transformation in algebra in which algebra is broadened so that it is no longer restricted to quantities, but is rather qualitative, under which the traditional algebra of quantities is subsumed. The '*combinatorial characteristic* then is the purely formal logic that Leibniz wishes to construct. There is a dialectic relation between algebra and logic here. It is through algebra, and especially its characteristic of grasping truth that the universal calculus of logic is first realised but once discovered this logic is a more generalised algebra on which algebra itself is founded. In philosophy we have an easy way of dissolving this apparent paradox. We can simply say that whereas logic has the ontological primacy over mathematics and particularly algebra, algebra has an epistemological primacy over logic. For our purposes it would be better to say that algebra, particularly modern abstract algebra is historically prior to modern symbolic logic.

Leibniz clearly anticipated that an algebra that was qualitative and quantitative could be realised:

> But this art can be and ought to be used not only when our concern is with formulas which express magnitudes, and with the solution of equations, but also when the involved key is to be developed for other formulas which have nothing in common with magnitude. The art of finding progressions and of establishing tales of formulas is also purely combinatorial, for these have a place not only in formulas expressing magnitude but in all others as well. For formulas can also

> be derived from them which express situation [*situs*] and the construction of lines and angles without considering magnitude. More elegant constructions can be discovered by this method, and more easily, than through the computing of magnitudes. With the help of combinatorial theorems […] it can be proved far more naturally than Euclid has done that the sides of triangles having equal angles are proportional. (1678, 193)

This passage also clearly states the logicist thesis that would be characteristic of at least one group of modern logicians, mainly philosophers like Frege and Russell, but also of mathematicians like Couturat.

Here is another account of algebra of logic as the art of combinations:

> […] the whole of algebra is an application to quantities of the *art of combinations*, or of the science of abstract forms, which is the universal characteristic and belongs to metaphysics. So the product of the multiplication of *a*+*b*+*c*+etc. by *l*+*m*+*n*+etc. is nothing but the sum of all the binary combinations which can be built out of the letters of the two series, and the product of the three series *a*+*b*+*c*+etc., *l*+*m*+*n*+etc., and *s*+*t*+*v*+etc. is the sum of all the ternary combinations which can be built from the three series of letters. Other forms will be produced from other operations. (1714, 670)

Leibniz is here very close to the invention of modern logic just two years before his death. Combinatory algebra allows us to go past the limitations of classical logic in many ways. First, it provides the algebra for truth functions of connectives, second it provides a logic of relations whereas classical logic is limited to subject–predicate propositions. It is interesting to note that Leibniz here identifies the art of combinations with metaphysics, whereas Boole in the quotation mentioned in section 1 of this paper (p. 2) claimed that 'we ought no longer to associate Logic and Metaphysics but Logic and Mathematics.' However, the two claims are quite compatible, because as we said earlier, Boole means by 'metaphysics' what is traditionally understood as metaphysics, having something to do with the world even if it is the world of conceptual schemes. However, for Leibniz 'metaphysics' is purely formal second order philosophy. So, his identification of 'metaphysics' is both with 'mathematics' as well as 'logic' both of which are also second order disciplines which are not about the world, so to speak. This is the point where I started this presentation.

3.3 Proofs and Identity

Leibniz perhaps made the greatest contribution to ontology since Aristotle as he extended the categories which presented the ontology of the world or of a conceptual scheme to syncategories which cut across categories and may be thought of as metacategories.[2] Identity is one of these syncategories. Leibniz had the insight that logical proofs are simply reductions to identities and identities are always self-identities of the form $x = x$. Here is an example:

Using the definitions:

(i) $2 = 1 + 1$
(ii) $3 = 2 + 1$
(iii) $4 = 3 + 1$

[2] This notion of syncategories comes from my teacher the late Professor Hector-Neri Castañeda in his course on Leibniz in 1977.

We can prove:

2 + 2 = 2 + 2	by principle of identity
2 + 2 = 2 + (1+1)	by definition (i)
2 + 2 = (2 + 1) + 1	association
2 + 2 = 3 + 1	by definition (ii)
2 + 2 = 4	by definition (iii)

(Kneale and Kneale 1962, 333)

We may start with a self-identity as in the above example and by using only definitions and substitution of identicals proceed to demonstrate the conclusion that we have to establish the truth of. Or we may follow the reverse procedure of starting with the conclusion and reducing it to a self-identity:

2 + 2 = 4	
2 + 2 = 3 + 1	by definition (iii)
2 + 2 = (2 + 1) + 1	by definition (ii)
2 + 2 = 2 + (1+1)	association
2 + 2 = 2 + 2	by definition (i)

If Leibniz is right in his claim that all proofs can be reduced to use of identity and definitions alone, then all of mathematics is reducible to logic which is the claim of logicism. Perhaps most mathematicians would be opposed to this reduction. Kneale and Kneale (1962, 334) cite the example of Fermat's last theorem as a counterexample to logicism. Until 1962 Fermat's last theorem had not been proven, the Kneales contend that if proofs were simply a matter of the use of only definitions and identities then surely the theorem would have been proven long ago. Today the theorem stands proven by Wiles. If Leibniz were alive today he would assert his logicism by reducing this theorem to his procedure of either beginning with a self-identity or by beginning with the conclusion and reducing it to a self-identity. Just because the example we have given above is elementary does not mean that all proofs using only identities and definitions will be so simple. The actual proof like that of Wiles which uses elliptical geometry may be very complex, but once a proof is found by a mathematician, it can be reduced by a logician to a proof that uses only identities and definitions.

However, Leibniz does not seem to be justified in claiming that only definitions and self-identity are used in his proof. The property of association is used. Also the principle of the substitution of identicals for identicals is used. These will have to be established. Furthermore, the implicit definition of 1 as the successor of 0 is used. Now, we will need the definition of 0 which may be one of the most difficult tasks. Alternatively we can have the axiom that 0 is a number and another axiom that the successor of a number is a number, thereby 1 is a number. All of this was not available to Leibniz. However, Leibniz was on very firm grounds to claim that whatever is established through a proof is totally analytic, that is, that it is proven by necessary truths like self-identity, axioms, definitions and rules of inference. The question then remains whether there is anything new under the sun, whether proofs of logic and mathematics can ever be ampliative when they are analytic and not constructive? Such would be the objection of intuitionists. Leibniz would respond that there is something ampliative about analytic proofs. Let us go back to our example. When we look at the starting point of $2 + 2 = 2 + 2$, we may not immediately see the statement of the conclusion $2 + 2 = 4$ in it. Or, conversely, we may not immediately see that $2 + 2 = 4$ is reducible to the self-identity $2 + 2 = 2 + 2$. If someone has doubt about this let us change it to an example of complex numbers used by Leibniz:

$\sqrt{1+\sqrt{-3}} - \sqrt{1-\sqrt{-3}} = \sqrt{6}$ (1702, 544)

Let us give a Leibnizian proof of this:

$$\sqrt{1+\sqrt{-3}} - \sqrt{1-\sqrt{-3}} = \sqrt{6}$$
$$(\sqrt{1+\sqrt{-3}} - \sqrt{1-\sqrt{-3}})^2 = 6$$
$$1 + \sqrt{-3} - 2(\sqrt{1+\sqrt{-3}})(\sqrt{1-\sqrt{-3}}) + 1 - \sqrt{-3} = 6$$
$$2 - 2(\sqrt{1+\sqrt{-3}})(\sqrt{1-\sqrt{-3}}) = 6$$
$$-2(\sqrt{1+\sqrt{-3}})(\sqrt{1-\sqrt{-3}}) = 4$$
$$(\sqrt{1+\sqrt{-3}})(\sqrt{1-\sqrt{-3}}) = -2$$
$$(\sqrt{1+\sqrt{-3}})(\sqrt{1-\sqrt{-3}}) = 4$$
$$1 - (-3) = 4$$
$$4 = 4$$

We have got our desired reduction to a self-identity. But looking at our starting equation the self-identity of $4 = 4$ though anticipated in it is hardly self-evident and hence the derivation does get somewhere.

Another way to put it is to make the philosophical distinction between the context of discovery and context of justification. Actual proofs in mathematics are of course not given in the manner that Leibniz has given them. In the actual process of a proof a mathematician does come upon something new, yet this proof can ultimately be reduced to that of either starting with or ending up with a self identity once the proof is found. Hence, in the context of discovery when a new proof is found as for Fermat's last theorem we have come upon something novel. However, in the justification of this proof itself we may be able to reduce it to an identity.

Another example is provided by John Woods in response to a paper on abduction that I was to present in Windsor, Ontario in June 2009:

> Suppose that we wanted to have an axiomatic proof of the completeness of some system S. Then "S is complete" would be the proof's target. Our task now would be to find premisses which together imply that conclusion. This is a premiss-search task. Premiss searches are cutdown problems. They are filtrations of up to indefinitely large spaces of options to smaller – sometimes unit – subsets. Roughly speaking, the filter's function is to pick out the options that best serve the searcher's current objective. In the case of an axiomatic proof of S's completeness, a successful cutdown filter will pick out *theorems* to serve as premisses. To simplify somewhat,[3] if the filter provides us with theorems $T_1 \ldots T_n$, and $\{T_1 \ldots, T_n\}$ entails "S is complete", then we have the desired proof. (2009, 2)

What seems to be at least implicit in this is that this type of proof is ampliative here in a backward sense in that something new comes out in the search for premises. If we were to reduce this proof to a Leibnizian proof of starting with an identity and ending up with the conclusion of the completeness theorem then the completeness theorem would be ampliative.

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[3] If the searcher's desire is to have an *elegant* proof of S's completeness, the filter will have to be more discriminating, aiming at the theorems minimally necessary for the desired entailment.

.4 Axiomatic approach including soundness and completeness proofs

Leibniz made a brave attempt to axiomatise the universal logical calculus in 'A Study in the Logical Calculus (early 1690's). Leibniz offers two axioms and two postulates:

Axiom 1: $B \oplus N \infty N \oplus B$

Axiom 2: $A \oplus A \infty A$

Postulate 1: Given any term whatever, something can be assumed to be diverse from it, and, if desired, disparate, or so that one is not in the other.

Postulate 2: Any number of terms whatever, such as A and B can be added together into one, composing $A \oplus B$ or L. (p. 372)

With the help of six definitions Leibniz goes on to state and prove 24 propositions. The inspiration of his axiomatisation is obviously Euclid's *Elements*. The postulates here seem to be more like Euclid's common notions. This was Leibniz's second attempt at constructing the propositional calculus. The entire 'A Study in the Logical Calculus' is provided below in Appendix 1.

How rigorous and how successful was Leibniz's attempt at providing an axiomatised logical calculus? This is perhaps for expert mathematicians to decide. Wolfgang Lenzen (2004) has expressed the Leibnizian logical calculus more formally in order to put forward his thesis that Leibniz's 'algebra of concepts' is isomorphic to Boole's algebra of sets (p. 9). A sketch of the formalisation is given in Appendix 2. Lenzen's formalisation of Leibniz is remarkable. He combines, besides the work of Leibniz in Appendix 1 all other works of Leibniz on logic and cumulatively builds up what can best be described as a modal predicate calculus with proofs of soundness and completeness, all given by Leibniz himself. Here is an example of How Lenzen formalises Leibniz's formulation:

Laws of L1	Formal version	Leibniz's version
CONT 1	$A \in A$	"*B* is *B*" (**GI**, §37)
CONT 2	$A \in B \wedge B \in C \to A \in C$	"[...] if *A* is *B* and *B* is *C*, *A* will be *C*" (**GI**, §19)
CONT 3	$A \in B \leftrightarrow A = AB$	"Generally '*A* is *B*' is the same as '*A* = *AB*' " (**GI**, §83)
CONJ 1	$A \in BC \leftrightarrow A \in B \wedge A \in C$	"That *A* contains *B* and *A* contains *C* is the same as that *A* contains *BC*" (**GI**, §35; cf. **P** 58, note 4)
CONJ 2	$AB \in A$	"*AB* is *A*" (**C**, 263)
CONJ 3	$AB \in B$	"*AB* is *B*" (**GI**, §38)
CONJ 4	$AA = A$	"*AA* = *A*" (**GI**, §171, Third)
CONJ 5	$AB = BA$	"*AB*∞*BA*" (**C**. 235, # (7))
NEG 1	$\overline{\overline{A}} = A$	"Not-not-*A* = *A*" (**GI**, §96)
NEG 2	$A \neq \overline{A}$	"A proposition false in itself is '*A* coincides with not-*A*' " (**GI**, §11)
NEG 3	$A \in B \leftrightarrow \overline{B} \in \overline{A}$	"In general, '*A* is *B*' is the same as 'Not-*B* is not-*A*' " (**GI**, §77)
NEG 4	$\overline{A} \in \overline{AB}$	"Not-*A* is not-*AB*" (**GI**, §76a)
NEG 5	$[\mathbf{P}(A)\wedge] A \in B \to A \notin \overline{B}$	"If *A* is *B*, therefore *A* is not not-*B*" (**GI**, §91)
POSS 1	$\mathbf{I}(A\overline{B}) \leftrightarrow A \in B$	"if I say '*A* not-*B* is not', this is the same as if I were to say [...] '*A* contains *B*' " (**GI**, §200).[16]
POSS 2	$A \in B \wedge \mathbf{P}(A) \to \mathbf{P}(B)$	"If *A* contains *B* and *A* is true, *B* is also true" (**GI**, §55)[17]
POSS 3	$\mathbf{I}(A\overline{A})$	"*A* not-*A* is not a thing" (**GI**, §171, Eighth)
POSS 4	$A\overline{A} \in B$	

(p. 15)

Furthermore, Lenzen shows how Leibniz came very close to complete axiomatised formulation of syllogistic logic. Here are the rules of syllogisms:

(Opp 1) $\neg\mathbf{A}(B,C) \leftrightarrow \mathbf{O}(B,C)$

(Opp 2) $\neg\mathbf{E}(B,C) \leftrightarrow \mathbf{I}(B,C)$

(Sub 1) $\mathbf{A}(B,C) \to \mathbf{I}(B,C)$

(Sub 2) $\mathbf{E}(B,C) \to \mathbf{O}(B,C)$

(Conv 1) $\mathbf{E}(B,C) \leftrightarrow \mathbf{E}(C,B)$

(Conv 2) $\mathbf{E}(B,C) \to \mathbf{O}(C,B)$

(Conv 3) $\mathbf{A}(B,C) \to \mathbf{I}(C,B)$

(Conv 4) $\mathbf{I}(B,C) \leftrightarrow \mathbf{I}(C,B)$.

(p. 56)

The following are the four basic valid forms of syllogism as discussed in section 2 above:

(Barbara) $\mathbf{A}(C,D) \wedge \mathbf{A}(B,C) \to \mathbf{A}(B,D)$

(Celarent) $\mathbf{E}(C,D) \wedge \mathbf{A}(B,C) \to \mathbf{E}(B,D)$

(Darii) $\mathbf{A}(C,D) \wedge \mathbf{I}(B,C) \to \mathbf{I}(B,D)$

(Ferio) $\mathbf{E}(C,D) \wedge \mathbf{I}(B,C) \to \mathbf{O}(B,D)$.

(pp. 56–7).

Lenzen now goes on to show that as valid forms in other figures can be reduced to these four valid forms in the first figure, we have an axiomatic representation of classical logic: 'Hence {BARBARA, CELARENT, DARII, FERIO, OPP 1, 2} constitutes an axiomatic basis for the theory of the syllogism' (p. 57). Leibniz was able to derive a proof of soundness quite easily, but he was not really able to provide a proof of completeness. To help Leibniz with the completeness, Lenzen goes on to sketch a more extensive axiomatic syllogistic logic, again derived from Leibniz himself, and then goes on to sketch the completeness proof (pp. 58–61).

.5 Modal and deontic logic

As discussed in 3.4 Lenzen believes that Leibniz more or less had a modal calculus at hand along with a possible world semantics. He sketches the following:

(NEC 1) $\Box(\alpha) \leftrightarrow \neg\Diamond(\neg\alpha)$

(NEC 2) $\neg\Diamond(\alpha) \leftrightarrow \Box(\neg\alpha)$.

(NEC 3) $\Box\alpha \rightarrow \Diamond(\alpha)$,

(NEC 4) $\neg\Diamond(\alpha) \rightarrow \neg\Box(\alpha)$.

(NEC 5) $\Box\alpha \rightarrow \Box\Box\alpha$.

(NEC 6) $\neg\Diamond\alpha \rightarrow \neg\Diamond\Diamond\alpha$.

(pp. 40–46)

Lenzen then goes on to show how Leibniz builds a deontic logic on his modal logic with the following:

(DEON 1) $O(\alpha) \leftrightarrow \Box_b(\alpha)$

(DEON 2) $E(\alpha) \leftrightarrow \Diamond_b(\alpha)^{70}$

(DEON 3) $F(\alpha) \leftrightarrow \neg\Diamond_b(\alpha)$

(DEON 4a) $O(\alpha) \rightarrow E(\alpha)$

(DEON 4b) $\neg E(\alpha) \rightarrow \neg O(\alpha)$

(DEON 5a) $O(\alpha) \rightarrow \neg F(\alpha)$

(DEON 5b) $F(\alpha) \rightarrow \neg O(\alpha)$

(DEON 6) $F(\alpha) \leftrightarrow O(\neg\alpha)$

(DEON 7) $O(\alpha) \leftrightarrow F(\neg\alpha)$

(DEON 8) $\neg O(\alpha) \leftrightarrow E(\neg\alpha)$

(DEON 9a) $\Box(\alpha) \rightarrow O(\alpha)$

(DEON 9b) $\neg O(\alpha) \rightarrow \neg\Box(\alpha)$

(NEC 7) $\Box(\alpha) \rightarrow \Box_b(\alpha)$.

(NEC 8) $\Box_b(\alpha) \leftrightarrow \neg\Diamond_b(\neg\alpha)$.

(NEC 9) $\Box_b(\alpha) \rightarrow \Diamond_b(\alpha)$

(DEON 10a) $\Box(\alpha) \rightarrow E(\alpha)$

(DEON 10b) $\neg E(\alpha) \rightarrow \neg\Box(\alpha)$

(DEON 11a) $E(\alpha) \rightarrow \Diamond(\alpha)$

(DEON 11b) $\neg\Diamond(\alpha) \rightarrow \neg E(\alpha)$

(DEON 11a) $E(\alpha) \rightarrow \Diamond(\alpha)$

(DEON 11b) $\neg\Diamond(\alpha) \rightarrow \neg E(\alpha)$

(DEON 12a) $O(\alpha) \rightarrow \Diamond(\alpha)$

(DEON 12b) $\neg\Diamond(\alpha) \rightarrow \neg O(\alpha)$

(DEON 13) $\neg E(\neg\alpha) \rightarrow E(\alpha)$.

(pp. 40–46)

As can be seen in (DEON 1) the subscript 'b' of the necessity operator indicates 'it is necessary for a virtuous person' and not unqualified 'it is necessary'. By (NEC 7) the unqualified 'it is necessary' implies 'it is necessary for a virtuous person', but the implication does not hold the other way. So, though ought does not imply is, is does imply ought, a powerful insight of Leibniz. Furthermore, the principle that if one is obligated to do something then one must be able to do it is also captured by (DEON 12a).

Lenzen remarks that the importance of Leibniz's development of modal logic is that 'By means of a simple, ingenious device, Leibniz transformed the algebra of concepts into an algebra of propositions' (p. 34). If Leibniz really did accomplish this it would justify Couturat's claim that Leibniz 'on certain points went beyond Boole himself'. For Boole failed to develop a propositional logical calculus from his algebra of logic in terms of classes. A second point where Leibniz went beyond Boole was his soundness and completeness proofs discussed in section 3.4. Leibniz could easily have developed an epistemic logic as well, but if Lenzen has not found this, then Leibniz must not have considered it.

.6 Limitations of Leibniz

Despite the remarkable achievements of Leibniz we cannot say that modern logic emerged in Leibniz. The main reason is the limitations of classical logic that he could not overcome. Yet, Leibniz came so close that often his personality is cited for the incompletion of his project: 'He was a universal genius who conceived many projects and made many beginnings but brought little to fruition.' (Kneale and Kneale 1962, 320) This is perhaps too harsh, and as a die hard fan of Leibniz I will come to his defence by quoting from another universal genius: 'For the beginning is thought to be more than half of the whole, and many of the questions we ask are cleared up by it.' (Aristotle, *Nicomachean Ethics*, 1098 b5) Aristotle of course is talking about making genuine beginnings. I have personally made many beginnings in my philosophical carrier but perhaps none of them is a genuine beginning. In the case of Leibniz he made a great number of genuine beginnings, and a universal algebra of logic was one of these beginnings. So, we can say that half of the emergence of modern logic took place with Leibniz. But of course we cannot call half a task the whole of the task, hence we will have to wait until Boole for the dawn of modern logic.

For a better account of why Leibniz could not complete the task that he so seriously began, let us return to Couturat. Couturat is aware that it would be an exaggeration to claim that modern logic emerged with Leibniz because that simply did not happen:

> Comment se fait-il alors qu'il n'ait pas réussi à constituer définitivement la Logique algorithmique, comme Boole l'a fait un siècle et demi après lui? C'est qu'entre tant d'essais et de projets divers, il n'a pas su discerner le meilleur, l'adopter et le développer systématiquement. Et il y à cela plusieurs raisons. D'abord, par un respect excessif pour la tradition, il tenait à justifier la subalteranation et la conversion partielle, et par elles les modes du syllogisme dont la Logique moderne a établi l'illégitimité. Ensuite, par égard pour l'usage de la langue, il n'a pas su définir avec précision la *portée existentielle* des propositions particulières et universelles. Enfin et surtout, il n'a pas eu l'idée de juxtaposer et de combiner entre elles l'addition et la multiplication logiques, et de les traiter simultanément. Or cela vient de ce qu'il se plaçait de préférence au point de vue de la compréhension; par suite, il ne considérait qu'un seul mode de combinaison des concepts : l'addition leurs compréhensions, et négligeait l'autre mode : l'addition de leurs extensions. C'est ce qui l'a empêché de découvrir la symétrie ou la réciprocité de ces deux opérations, qui se manifeste par les formulas de DE MORGAN[4], et de développer le calcul de la négation, qui repose sur ces formulas. C'est aussi ce qui l'a amené à croire (à tort) que les relations d'extension obéissent aux memes lois que les relations de

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[4] $(a + b)' = a' b'$, $(ab)' = a' + b'$

compréhension, et à les considérer comme réversibles, en changeant simplement le sens de l'inclusion[5]. L'échec final de son système est donc extrêmement instructif, car il prouve que la Logique algorithmique (c'est-à-dire en somme la Logique exacte et rigoureuse) ne peut pas être fondée sur la considération confuse et vague de la compréhension; elle n'a réussi à se constituer qu'avec Boole, parce qu'il l'a fait reposer sur la considération exclusive de l'extension, seule susceptible d'un traitement mathématique[6].

How is it then that he did not succeed to definitively constitute the algorithmic logic, as Boole was able to do one and a half century later? It is between his too many essays and diverse projects, that he did not know to discern the better, to adopt and develop it systematically. And there are many such reasons. To begin with, due to excessive respect for tradition, he held to justify subalernation and partial conversion, and by them the modes of syllogisms that modern logic has established as illegitimate. Next, paying regard to the usage of language, he did not know how to define with precision the existential scope of particular and universal propositions. Finally and above all, he did not have the idea of juxtaposing and combining between both the addition and the multiplication logics, and treating them simultaneously. In fact this comes from giving preference to the point of view of comprehension; as a result, he only considered the mode of the combination of concepts: the comprensions of addition, and neglected the other mode: the addition of their extensions. It was this that impeded the discovery of symmetry or reciprocity of these two operations which are manifested by the formulas of De Morgan[4], and developing the calculus of negation, which is founded on these

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[5] En réalité, en vertu du *principe de dualité*, on peut renverser le signe d'inclusion (ou bien en changer le sens), mais à la condition de remplacer partout les signes de l'addition et de la multiplication l'un par l'autre (ou de permuter leurs significations). Mais il n'est pas permis de changer le sens des inclusions sans changer en meme temps celui des deux opérations, et c'est justement ce que Leibniz croyait possible.

In reality, in virtue of the *principle of duality*, one can reverse the sign of inclusion (or better change its sense), but due to the condition of replacing all the signs of addition and multiplication one by the other (or of switching around their significations). But he did not permit to change the sense of inclusion without at the same time changing both these operations, and it is justifiably this that Leibniz believed as possible. (my translation)

[6] On remarquera que, dans tous ces essais de Calcul logique, Leibniz est resté continé dans le domaine de la Logique classique, qui est celui des jugements de prédication, de la forme : << A est B. >> Mais il convient de rappeler ici qu'il a eu tout au moins l'idée d'une Logique plus générale, qui étudierait d'autre relations entre les concepts que la relation d'inclusion (ou, au point de vue grammatical, d'autres copules que le verbe *être*). D'une part, sous l'influence de Jungius, il avait entrevu des formes de raisonnement *asyllogistiques*, comme les inférences du droit à l'oblique, et l'inversion des relations (Chap. III, # 15). D'autre part, comme mathématicien, il avait aperçu qu'il y a entre les objets de la pensée bien d'autres relations que celle d'inclusion, et il se proposait de construire des algorithmes appropriés à chacune d'elles (Chap. VII, #8). Mais tout cela semble être resté à l'état de rève ou d'ébauche. Ce n'est qu'au XIX[e] siècle que s'est constituée une Logique vraiment universelle, la Logique des relations (dans laquelle renter la Logique classique, meme généralisée sous sa forme algorithmique) par les travaux de De Morgan, de Peirce et de Schröder.

One will notice, that in all these essays on the logical Calculus, Leibniz continuously stays in the domain of classical logic, which is in the judgements of predication, of the form: <<A is B.>> But he found convenient to call back this idea that he had of more or less of a most general logic, that will study the other relations between these concepts other than the relation of inclusion (or, to the point of surface grammar, of other copulas than the verb *to be*). On the one hand under the influence of Jungius, he had caught a glimpse of forms of *asyllogistic* reasoning, like the inference of the straight line from the oblique, and the inversion of relations (Chapter III, #15). On the other hand, as a mathematician, he had noticed that among the objects of thought there are relations other than that of inclusion, and he proposed to construct the appropriate algorithms for each of these (Chapter VII, #8). But all of this seems to rest on the state of a dream or of a sketch. It is only in the 19[th] century that a truly universal calculus, the logic of relations (in which classical logic comes back, ever more generalised under its algorithmic form) by the works of De Morgan, Peirce and Schröder. (my translation)

> formulas. This is also what led him to believe (mistakenly) that the relations of extension obey the same laws as the relations of compression, and considered them as reversible, in simply changing the sense of inclusion[5]. The final setback of his system is therefore extremely instructive, because it proves that the algorithmic logic (that is to say in sum the exact and rigorous logic) cannot be founded on the confused and vague consideration of comprehension; it could not succeed without Boole, because he built them on the exclusive considerations of extension, which alone is susceptible to a mathematical treatment. (my translation)

In sum, Leibinz's inability to come up with the full-fledged development of modern logic was due to his inability to transcend all the limitations of classical logic. The last sentence clear states that modern logic would have to wait until Boole for its emergence, and hence it is the perfect transition to our next section.

4. The Emergence of Modern Logic in Boole's Algebra of Logic

I say nothing new with the title of this section but merely announce my agreement with what seems to be the consensus among historians of logic, that modern logic can be said to have emerged with Boole's algebra of logic in the middle of the nineteenth century. I have jumped one and a half centuries from Leibniz to Boole, consequently skipping big name philosophers such as Berkeley, Hume, Reid, Wolff, Kant, Mill, Bolzano and Hegel. Earlier, I have also more or less ignored big name philosophers of the seventeenth century such as Descartes, Malebranche, Spinoza and Locke, whereas I have claimed some significant contribution of Hobbes. I have not done this due to paucity of time, as I am attempting to offer as comprehensive an account of all the essential factors that were responsible for the emergence of modern logic. Rather, I believe that these philosophers' contributions to the development of modern symbolic logic were minimally significant even though their contributions to the philosophy of logic, especially that of Kant and Bolzano may be significant. On the other hand the contributions of mathematicians to the emergence of modern logic during these 150 years were infinitely more significant, both of big name mathematicians and the not so famous mathematicians. I will hence turn in the next two subsections 4.1 and 4.2 to the transition of mathematics from being restricted to the domain of quantity to an abstraction where it is applied to a more generalised domain of which quantity is a subclass.

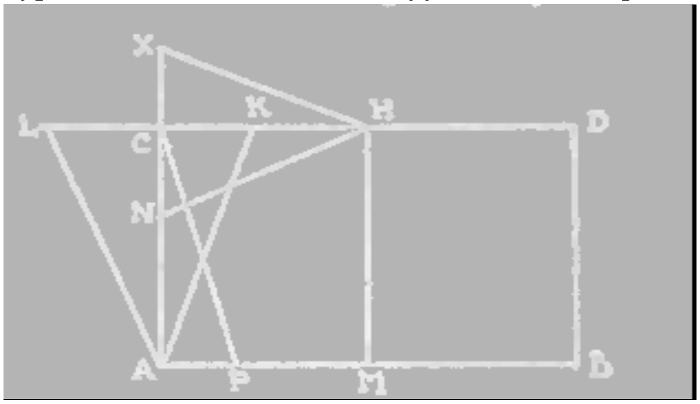
To continue with our transition from section 3 and to justify why we are now turning to the development of mathematics in the late eighteenth and early nineteenth centuries let me restate the quotation from Kneale and Kneale (1962, 378):

> Although Leibniz had put forward a number of brilliant suggestions, it was not possible to make sure progress until mathematics had developed so far that the sort of abstraction he desired seemed natural and easy. When logic was revived in the middle of the nineteenth century, the new vigour came from mathematicians who were familiar with the progress of their own speciality, rather than from philosophers who were occupied with the controversies of idealism and empiricism.

4.1 Non-Euclidean Geometry and topology as abstractions from quantity

One of the most interesting episodes in the history of geometry was Giovani Girolamo Saccheri's (1667–1733) attempt to prove Euclid's fifth postulate in *Euclides*

Vindicatus (1733). In a *reductio* set up Saccheri sets out to prove the right angle hypothesis of Euclid in the Khayyam–Saccheri quadrilateral:



If angles *A*, *B* and *D* are right angles, it must be demonstrated that *AC* is the only line that can be drawn to make the quadrilateral in which angle *C* is also a right angle. The *reductio* argument takes one by one, the possibility of *AK* as a line which can be drawn to complete the quadrilateral in which ∠*AKD* is obtuse; and the possibility of *AL* as a line that can be drawn to complete the quadrilateral in which ∠*ALD* is acute. In proving these latter two false, Saccheri comes very close to developing non-Euclidean geometries of the hyperbolic and elliptical kinds as he develops the possible theorems base on these alternate hypotheses. Of course Saccheri would never have dreamt that he was inventing non-Euclidean geometries as his purpose was to prove Euclid's fifth postulate by proving that these two alternatives were false. In his proof against the acute angle hypothesis there is a flaw in that Saccheri assumes that a line could be extended infinitely as this does not happen in elliptical geometry. (Kneale and Kneale 1962, 381) If Saccheri means 'a Euclidean line' then he begs the question. If Saccheri had been successful in his proof then he would have proven the consistency of Euclidean geometry as the other four postulates had already been proven. His failure to actually have proven what he thought he had proven does not of course imply that Euclidean geometry is not consistent, but what it does imply is that if it is consistent then so could non Euclidean geometries be consistent but this would have to be established on its own. (Ibid. p. 382) Euclidean geometry appeals to the geometrical intuition of space and many believe that it is still the correct geometry for three dimensional space that we live in whereas non-Euclidean geometries apply to two and more than three dimensional spaces. Since these spaces do not involve the three dimensional spatial intuition that humans have, it involves a level of abstraction from what is normally thought of as quantity in geometry. This brought out the realisation that what was important in Euclidean geometry and what is important in all axiomatic systems is that given the axioms the rest of the system follows through purely logical inferences, as Kneale and Kneale put it:

> When one proposition follows logically from another, it should be possible to formulate the two propositions in such a way that their relation can be seen to depend on their form alone, that is to say, on their logical structure as opposed their special subject-matter. (p. 384)

Euclidean geometry was developed since Saccheri by Gauss (1777–1855), Lobachevski (1792–1856), Bolyai (1802–1860) and Riemann (1826–1866). What the development of non-Euclidean geometries showed was that alternative sets of axioms could be used to build alternative formal systems of geometry each of which was consistent. This brought into question whether the axioms were true in the traditional notion of 'true' as corresponding to facts or whether they were true by convention. If the latter is the case then we would have to rethink the distinction between mathematical demonstrations and dialectical arguments, as it seems that in both we are now starting with the assumption that the axioms are true rather than from the truth of the axioms. I will not enter this controversy now, and emphasise again that the significance of the development of non Euclidean geometries for modern logic was that there could be an axiomatic geometry that did not rely on the intuition of space, which we could call purely formal geometry following from the set of axioms that are provided for it.

The other development was the emergence of topology as qualitative geometry by Euler (1707–1783), Mobius (1790–1868), Listing (1808–1882), Jules Henri Poincaré (1854–1912) and Oswald Veblen (1880–1960). In order to understand the level of abstraction that topology introduced let us define two notions as Kneale and Kneale do:

> It two different interpretations of any kind for the extra-logical signs in a set of axiom formulae both yield truths and the systems of things described by the formulae in these interpretations are so related that for every item in the one there is a single corresponding item in the other, the two interpretations are said to be isomorphic. If, further *all* the results of the possible interpretations (or models, as they are sometimes called) for a certain set of axiom formulae are isomorphic one to another, the set of axioms is said to be categorical or monomorphic. (p. 387)

Topology as qualitative geometry studies all 'properties that are invariant under the operations of a group' (ibid. p, 389). Whereas the axiom systems of lower level geometry, metrical geometry, are monomorphic, the axioms systems of topology as descriptive geometry may or may not be monomorphic. In Vebelen's descriptive geometry for example the axiom set is monomorphic with respect to *betweenness* but polymorphic with respect to *congruence*. Hence, from the same topological geometry alternative metrical geometries can be derived. In this way isomorphisms among Euclidean and non-Euclidean geometries can be established at the level of topology within a monomorphic set of axioms. This universality of topology relative to metrical geometry set the ground for a universal logical calculus.

4.2 Development of algebra not restricted to quantity

From the end of the eighteenth century to the early nineteenth century algebra also took a turn towards the more abstract. Group theory was developed by Cauchy (1789–1857), Abel (1802–1829), Galois (1811–1832), Cayley (1829–1895) and others. Also, the concept of numbers was also evolving towards greater abstraction. William Hamilton (1805–1865) came up with the hyper-complex numbers quaternions of the form $x + yi + zj + wk$, where i, j and k represent rotations in three mutually perpendicular planes. Unlike complex numbers these numbers do not satisfy all the general rules of algebra, as $ij = -ji$. (Kneale and Kneale 1962, 398). A more generalised algebra was hence required.

The development during this period that was essential for Boole's algebra of logic was that of symbolic algebra. As we have already seen in Leibniz the art of combination in algebra is more concerned with the laws of combination itself and not so much with the objects of combination. George Peacock (1791–1858) distinguished between arithmetical and symbolic algebra. (Peckhaus 2009, 164). Peacock claims that the operations in symbolic algebra must be open to interpretations other than that in arithmetic:

> In the first transition from Arithmetic to Algebra, […] in the very first applications of such operations, the mere use of general symbols renders the proper limitation of their values, […] thus $a - (a + b)$ would obviously express an impossible operation in such a system of Algebra; but if $a + b$ was replaced by a single symbol c, the expression $a - c$, though equally impossible with $a - (a + b)$, would cease to express it. The assumption however of the independent existence of the signs + and – removes this limitation, and renders the performance of the operation denoted by – equally possible in all cases: and it is this assumption which effects the separation of arithmetical and symbolical Algebra, and which renders it necessary to establish the principles of this science upon a basis of their own […] It […] makes it necessary to consider symbols not merely as the general representatives of numbers, but of every species of quantity, and likewise to give a form to the definitions of the operations of Algebra, which must render them independent of any subordinate science […] in framing the definitions of algebraical operations, to which symbols thus affected are subjected, we must necessarily omit every condition which is in any way connected with their specific value or representation: in other words, the definitions of some operations must regard the laws of their combination *only*: thus the operations denoted by + and – must regard the *affection* of symbols (with their + and –, whether accompanied or not by any other signs of affection which they are capable of receiving) by them, according to an *assumed* law for the *concurrence* of those signs […] Again, in order that such operations may possess an invariable meaning and character, […] when any number of such operations are to be performed and of symbols to be combined by means of them, we shall suppose the results to be the same, in whatever order those operations succeed each other. (Peacock 1830, pp. viii–x)

Peacock hence establishes the primacy of combinations over what they combine in the new symbolic algebra.

However, Peacock is not quite able to divest pure algebra of quantity. This task is accomplished by Duncan Gregory who defined symbolic algebra as 'the science which treats of the combination of operations defined not by their nature, that is by what they are or what they do, but by the laws of combination to which they are subject' (1840, 208).[7] Hence, even if by nature the operations of + and – presuppose quantities, in the new symbolical algebra + and – are defined only by the laws of combination. It seems that Leibniz's dream came true, and this is exactly what Boole needed to build his algebra of logic in which + and – would have purely combinatorial meaning independent of what was being combined.

Boole and his immediate predecessors were interested in logic but their main concern in the development of symbolical algebra was a reform in algebra itself towards greater abstraction to complement similar moves in geometry discussed above. Though Peacock, Gregory and Boole were British there was a parallel, independent and similar development of symbolical algebra and from that of the algebra of logic by mathematicians such as Hermann Grassmann (1809–1879) and Robert Grassman (1815–1901). The British as well as the Germans mentioned were concerned with 'a reform of

[7] I have got this quotation of Gregory from Peckhaus 2009, 164.

mathematics by establishing an abstract view of mathematics which focused not on mathematical objects like quantities but on symbolic operations with arbitrary objects. The reform of logic was only secondary.' (Peckhaus 2009, 175)

Most historians of mathematics and logic claim that the developments of symbolical logic and the algebra of logic in Germany though more or less parallel in the first half of the eighteenth century were not interactive and did not borrow from each other. If this is true it brings out the important feature that there is was an internal move from the inside of algebra towards greater abstraction which led to the definition of 'combination' independent of the objects of combination but subject only to the laws of combination. This is important for realising why Leibniz could not and Boole could invent modern logic as it was the history of algebra reaching a peak that was not available to Leibniz.

Augustus DeMorgan (1806–1871) made a distinction between 'technical algebra' and 'logical algebra', the former was the art of using symbols under specified rules and the latter was the science of giving meaning to primary symbols and interpreting subsequent results (Peckhaus, 169). Logical algebra then rather than technical algebra is the universal calculus which can now be called the 'algebra of logic' hence fulfilling Leibniz's dream.

4.3 Boole's pathbreaking work: *The Mathematical Analysis of Logic*

Boole being a mathematician was greatly influenced by Peacock and Gregory's development of symbolical algebra and wanted to develop such an algebra which would not be restricted to quantity even further. Some historians of logic hence claim that Boole was more concerned with the development of symbolical algebra than of symbolic logic and the development of symbolic logic in *The Mathematical Analysis of Logic* 1847) was more to provide an example of the use of symbolical algebra than for the development of logic itself:

> Although Boole's logical considerations became increasingly philosophical with time, aiming at the psychological and epistemological foundations of logic itself, his initial interest was not to reform logic but to reform mathematics. He wanted to establish an abstract view on mathematical operations without regard to the objects of these operations. When claiming "a place among the acknowledged forms of Mathematical Analysis" (1847, 4) for the calculus of logic, he didn't simply want to include logic in traditional mathematics. The superordinate discipline was a *new* mathematics. (Peckhaus 2009, 166)

In this section part of my purpose is to unfold a narrative in which we see that Boole was very much aware even in this early work of the need to reform logic and this played as great a role in this path breaking work as did his desire to create a new mathematics. I have already quoted above that Boole wanted to divorce logic from metaphysics and wed it to mathematics (1847, p. 13). Kneale and Kneale state:

> The idea that algebraic formalae might be used to express logical relations occurred to him first when he was still in his teens […] But the renewed interest in logic which let him to write this little book in 1847 was due to the […] controversy in which Sir William Hamilton of Edinburgh claimed priority in adoption of the doctrine of the quantification of predicates, charged Augustus De Morgan with plagiarism, […] (1962, 404)

As my narrative unfolds we shall see that in this work Boole was not only consciously developing logic for the sake of logic, but also had keen insights on the history of logic, due to which he could show how the development of modern algebra could help logic transcend some of the difficulties that had retarded the growth of logic due the limitations of classical logic. If Boole's point was simply to establish logic as an example of a symbolical algebra which does not have objects of quantity, then he would not have made the efforts in the work that he does to show the intricate manner in which algebra could improve the understanding of traditional logic, and at one point he even claims that the algebra of quantity can come in to the aid of logic.

4.3.1 *The algebraic combinatorial analysis of logic*

In the first paragraph of the Introduction Boole explicitly states the characterisation of symbolical algebra to satisfy Leibniz's dream that I have discussed in section 4.2:

> They who are acquainted with the present state of the theory of Symbolical Algebra, are aware that the validity of the processes of analysis does not depend upon the interpretations of the symbols which are employed, but solely on the laws of their combination. Every system of interpretation that does not effect the truth of the relations supposed, is equally admissible, and it is thus that the same process may, under one scheme of interpretation, represent the solution of a question on the properties of numbers, under another, that of a geometrical problem, and under a third, a problem of dynamics or optics. (Boole, 1847, 3)

Boole is hence in search of the laws of combination that could be interpreted differently to generate analysis or geometry or dynamics. Isn't what Boole is looking for then not a universal calculus that Leibniz was looking for? There is almost a consensus among scholars that Boole did not become aware of Leibniz's attempt at a universal calculus for logic until much later. I have no reason not to trust the scholars or Boole's honesty, but it is really difficult to swallow this. What is not clear is that Boole takes logic to be this universal calculus of combinations or with this universal combinatory calculus would be pure mathematics with logic as branch of it. This ambiguity is reflected on the very next page:

> We might justly assign it as the definitive character of true Calculus, that it is a method resting upon the employment of Symbols, whose laws of combination are known and general, and whose results admit of a consistent interpretation. That to the existing form of Analysis a quantitative interpretation is assigned, is the result of the circumstances by which those forms were determined, and is not to be construed into a universal condition of Analysis. It is upon the foundation of this general principle, that I propose to establish the Calculus of Logic, and that I claim for it a place among the acknowledged forms of Mathematical Analysis, regardless that in its object and in its instruments it must as present stand alone. (ibid, 4)

The majority of this passage supports logicism, the thesis that logic is the foundation for all of mathematics, but the last two lines introduce a tension. It seems that on the one hand Boole wants to say that logic is the foundation for all of mathematics but on the other hand he wants to say that logic is a branch of mathematics. No wonder that Peckhaus who supports the latter as the correct state of mind of Boole at the time quotes only this last part without quoting the first part. I think we have a sort of a chicken and egg paradox here which can be easily resolved in a similar fashion that I have dealt with the appearance of a predicate logic before the appearance of propositional logic in section

2. If Boole could peak ahead reading through his own later works, and those of Frege, and Whitehead and Russell, he would resolve the present tension in favour of the logicist thesis. The paradox of logicism is that even though logic is the foundation of mathematics, the development of logic always lags behind that of mathematics; particularly in 1847, whereas symbolical algebra was already established, symbolic logic had not even properly begun in a formal manner. But once this was done, by the end of this very work of Boole, then this symbolic logic would become the foundation for symbolical algebra as well as other branches of mathematics. However, to establish this symbolic logic formally Boole would use the works of his immediate predecessors like Peacock and Gregory on symbolical algebra to construct symbolic logic.

This dialectic continues throughout the Introduction as Boole claims at one point that logic is applied mathematics (p. 10). He also wants to derive the theorems of logic just as the theorems of mathematics are derived (p. 6). But near the end of the Introduction after rescuing logic from metaphysics and placing under the partnership of mathematics he claims:

> Should any one after what has been said, entertain a doubt upon this point, I must refer him to the evidence which will be afforded in the following Essay. He will there see Logic resting like Geometry upon axiomatic truths, and its theorems constructed upon the general doctrine of symbols, which constitutes the foundation of the recognised Analysis. […] Logic not only constructs a science, but also inquires into the origin and the nature of its own principles,—a distinction which is denied to Mathematics. (ibid., 13)

This will definitely be welcome by logicists. First, Boole says that the logic he will construct can serve as the foundation of Analysis, and then he establishes that logic cannot be founded on any branch of mathematics since unlike mathematics it inquires into the origin of its own principles. It can hence serve as the foundations for any branch. However, Boole does not say that logic can serve as a foundation for all mathematics. Rather, he sees logic as a parallel to geometry. There is a remarkable parallel in Frege who claims logicism in arithmetic but not in geometry, but Frege's reason is that geometry requires spatial intuition and arithmetic does not require any intuition. Boole's reason seems to be that Geometry is axiomatic and the logic he builds will also be axiomatic. This is very interesting because as arithmetic had not yet been axiomatised Boole appeals to geometry in order to attempt an axiomatisation of logic. But as I will argue later the actual axiomatisation of logic was more or less simultaneous with that of arithmetic and set theory which goes a long way in supporting the logicist thesis.

However, in the next paragraph the dialectic takes a turn:

> The application of this conclusion to the question before us is clear and decisive. The mental discipline which is afforded by the study of Logic, *as an exact science*, is, in species, the same as that afforded by the study of Analysis. (ibid., 13)

If logic and analysis are both species then the genus they are the species of is mathematics so that logic is again declared as a branch of mathematics and the hopes of logicism to find a supporter in Boole are dashed. Boole goes on:

> Is it then contended that either Logic or Mathematics can supply a perfect discipline to the Intellect? The most careful and unprejudiced examination of this question leads me to doubt whether such a position can be maintained. The exclusive claims of either must, I believe, be abandoned, […] it is one thing to arrive at correct

> premises, and another thing to deduce logical conclusions, and that the business of life depends more upon the former than upon the latter.

Hence, at the end of the paragraph not only does Boole not accept logicism here but announces that logicism could never succeed as arriving at the correct premises is in the domain of mathematics and not in the domain of logic. In the first part of the paragraph Boole promotes a parallelism between mathematics and logic. So that we can conclude the Introduction by noting that all through the writing of the Introduction from the beginning to the end Boole is very consciously thinking of logic. Hence, it is unlikely that he wants to develop algebraic logic merely as a branch of mathematics, but as a new and radical logic itself.

4.3.2 ***The algebra of classical logic***

In the first section entitled FIRST PRINCIPLES Boole lays down the foundations of the algebra of logic which is a logic of classes. 1 represents the universe; X, Y, Z, represent members of classes; x, y, z, are elections which pick out members from the classes. The three combinatory laws are: (1) $x(y + z) = xy + xz$ (distributive) (2) $xy = yx$ (commutative); and (3) $x^2 = x$ (index). (3) is obviously the one that does not hold in ordinary algebra. (ibid., p. 15) Boole claims these three laws along with the axiom that equivalent operations performed on equivalent subjects produce equivalent results, constitute the axiomatic foundation on which all of logic can be built. (ibid., p. 18) The rest of the section goes on to the symbolization of propositions of classical logic that is summarized in a table:

TABLE.

The class X	x	
The class not-X	$1 - x$	
All Xs are Ys All Ys are Xs	$x = y$	
All Xs are Ys	$x(1 - y) = 0$	
No Xs are Ys	$xy = 0$	
All Ys are Xs Some Xs are Ys	$y = vx$	vx = some Xs $v(1 - x) = 0.$
No Ys are Xs Some not-Xs are Ys	$y = v(1 - x)$	$v(1 - x)$ = some not-Xs $vx = 0.$
Some Xs are Ys	$v = xy$ or $vx = vy$ or $vx(1 - y) = 0$	v = some Xs or some Ys vx = some Xs, vy = some Ys $v(1 - x) = 0$, $v(1 - y) = 0.$
Some Xs are not Ys	$v = x(1 - y)$ or $vx = v(1 - y)$ or $vxy = 0$	v = some Xs, or some not-Ys vx = some Xs, $v(1 - y)$ = some not-Ys $v(1 - x) = 0$, $vy = 0.$

(ibid., 25)

It seems like the same old story? Where is modern logic? Are we again in the same boat like Leibniz that we cannot escape the limitations of classical logic? These are tricky questions. Classical logic was still the only formal logic available to Boole, yet he realised the limitations of it perhaps even more than Leibniz. What he is trying to do in this essay is to construct a new logic, but he is going to do it by first giving an algebraic interpretation of classical logic, and when modern logic emerges, and I claim that this happens at the end of this work, then classical logic can be discarded if we so desire. What is new, what no one before him was able to do so clearly is to represent categorical statements as explicit algebraic equations.

In the next section entitled ON THE CONVERSION OF PROPOSITIONS the discussion of classical logic continues where the rules of conversion are explained with the help of algebraic equations.

In the next section OF SYLLOGISMS, the valid syllogisms of classical logic are arrived at algebraically by multiplying equations and eliminating *y* which represents the traditional middle term. The rule of elimination is specified as follows:

$$ay + b = 0$$
$$a' y + b' = 0$$

When *y* is eliminated this reduces to:

$ab' - a'b = 0$ (ibid., 32)

Here is how Boole captures the valid mood bArbArA in the first figure:

$y(1 - x) = 0$ or $(1 - x)y = 0$

$z(1 - y) = 0$ or $-zy + z = 0$

Now, let us multiply the two equations eliminating the y:

$(1 - x)\, z = 0$

By commutation we get:

$z(1 - x) = 0$. (ibid., 34)

Which is the desired AAA in the first figure.

Now let us try AA in the second figure. Does it yield a conclusion?

$x(1 - y) = 0$ or $-xy + x = 0$

$z(1 - y) = 0$ or $-zy + z = 0$

Now, let us multiply the two equations eliminating the y:

$-xz + xz = 0$

$0 = 0.$

Whenever the result is 0 = 0, then no conclusion can be reached (ibid., 38)so AA in the second figure does not yield a conclusion, which means all of AAA, AAE, AAI and AAO are invalid in the second figure.

Now, let us consider IO in the first figure. Does it yield a conclusion?

$vy = vx$ or $vy - vx = 0$ or $vy - vx = 0$

$v = z(1 - y)$ or $v - z(1 - y) = 0$ or $zy + (v - z) = 0$

Now, let us multiply the two equations eliminating the y:

$v(v - z) + vz = 0$

$vv - vz + vz = 0$

$v = 0$

So we have another type of invalid syllogisms in which the conclusion does not reduce to 0 = 0. Boole claims that this distinction of the two types of syllogisms where a valid conclusion is not derived is a purely mathematical one but it points out an important distinction in classical logic. In the second type which don't reduce to 0 = 0 there is no virtual middle term as a medium of comparison (ibid., 40). Hence Boole concludes:

> I am not aware that the distinction occasioned by the presence or absence of a middle term, in the strict sense here understood, has been noticed by logicians before. The distinction, though real and deserving attention, is indeed by no means an obvious one, and it would have been unnoticed in the present instance but the particularity of its mathematical expression. (ibid., 41)

This is a classic example of how the mathematisation of logic betters logic so that we can have a better understanding of language and the world. Again, a Leibnizian dream is realised by Boole and modern logic is emerging.

Boole goes on with more generalized equations for representing syllogisms and by the time we reach the end of this section on page 47, some modern logicians with dislike for classical logic may be getting impatient as we have finished 57% of the book and most of the work so far has been on the algebrisation of classical logic. So where is modern logic? I request your patience as Boole would have requested the patience of his readers. We must remember that Boole is not necessarily creating modern logic as we know it, but rather as the title of my presentation claims, modern logic is emerging in

Boole, perhaps unknown to Boole himself. A very important transition is taking place in these pages from 40 to 48. Boole realises the limitations of classical logic, such as the order of the premises, and also being restricted only to subject–predicate propositions. In the generalized algebraisation of syllogisms Boole is attempting to show algebraically that the order of premises is not important as classical logic claims to be, which is one of the features of modern logic. Hence, the final generalised representation of premises that he offers is:

$a + bx + cy + dxy = 0,$
$a' + b'z + c'y + d'zy = 0.$ (ibid., 47)

4.3.3 *The emergence of the algebra of propositional logic*

In the next section entitled OF HYPOTHETICALS Boole makes the crucial turn to propositional logic and this really is the emergence of modern logic as what has been missing all along, a propositional calculus, can now be developed using algebra. This is the monumental contribution of mathematics, algebra and Boole to logic. He turns to the discussion of conditionals. First he presents conditionals as what appear to be in terms of classes as in syllogistic logic:

If A is B, then C is D,
But A is B, therefore, C is D.

But then he expresses it in terms of propositions without reference to classes:

If X is true, then Y is true,
But X is true, therefore, Y is true.
(ibid., 48)

I have centred and these and changed the font to represent exactly what is going on in the book. I do not know whether this was intentional on Boole's part or a printing error or selection. Whatever be the case it does a perfect job of making the point that where the small font ends and where the big font begins is the emergence of modern logic. It is also interesting to note that the first example of a conditional argument given here is that of *modus ponens*. As modern logic would develop in its axiomatised form *modus ponens* would become the preferred single rule of inference by Frege and others.

Here is what Boole says about this remarkable transition:

> Thus, what we have to consider is not objects and classes of objects, but the truths of Propositions, namely, of those elementary Propositions which are embodied in the terms of our hypothetical premises. (ibid., 48–9).

We can embalm page 48 as the turning point where the long awaited transition from classical logic to modern logic occurs.

Boole calls the propositions not involving classes 'elementary propositions' (ibid., 49). Using 0 for false and 1 for true Boole now comes up with the possibilities for truth tables (ibid., 50–1) and goes on to define conjunction and disjunction (both exclusive and inclusive) and conditional truth functionally (ibid., 52–4). As truth values are algebrised we see again how mathematics can help us reach important insights in logic. These equations can be used for understanding truth functionality in a way that may not be understood without mathematics. Let us consider the equation for the exclusive disjunction 'Either X is true or Y is true'. Boole expresses this as: $x - 2xy + y = 1$, which is acquired from what we may call the second and third row of the truth table: $x(1 - y) +$

$y(1 - x) = x - xy + y - xy = x - 2xy + y$, and this must be true, so we set it equal to 1 as $x - 2xy + y = 1$. Now, since $x^2 = x$. We get: $x^2 - 2xy + y^2 = 1$. Which reduces to $(x - y)^2 = 1$; $x - y = \pm 1$. This is actually the case for when x is true having the value of 1, then y must be false having the value 0 so that the equation is satisfied. Similarly, when x is false, having the value 0, then y is true, having the value 1 to satisfy the equation. {ibid., 55} Hence, we see here a clear example from the inside of Boolean algebra how a simple algebraic operation, but without regard to quantity as the rule $x^2 = x$ is not a rule of ordinary algebra can lead to a clear definition of a logical operation like disjunction.

Boole next goes on to give algebraic formulations of *modus ponens* {$x(1 - \mathrm{y}) = 0$, $x = 1$, $\therefore\ 1 - y = 0$ or $y = 1$}, *modus tollens* { $x(1 - y) = 0$, $y = 0$, $\therefore\ x = 0$}, disjunctive syllogism (exclusive) {x + y – $2xy = 1$, $x = 1$, $\therefore\ y = 0$}, disjunctive syllogism (not exclusive) {$x + y$ – xy = 1, x = o, $\therefore\ y = 1$}, simple constructive dilemma {$x(1 - \mathrm{y}) = 0$, $z(1 - y) = 0$, $x + z - 2xz = 1$, $\therefore\ y = 1$}, composite constructive dilemma {$x(1 - \mathrm{y}) = 0$, $w(1 - z) = 0$, $x + w - xw = 1$, $\therefore\ y + z - yz = 1$}, and complex destructive dilemmas of two kinds. (ibid., 56–7). These representations will surely make the fans of natural deduction very happy. I may say that for the purposes of this winter school I shall remain a closet natural deduction fan.

4.3.4 *The role of functions in logic*

In the next section entitled PROPERTIES OF ELECTIVE FUNCTIONS Boole claims at the end that mathematicians will understand this much better than logicians, not being a mathematician and neither a genuine logician I will hence not even venture into the details of this section. A point that must be noted however is that Boole uses the mathematical notion of functions for the algebra of logic. Hence, he anticipates Frege's later use of function and argument to represent propositions. Even though Boole is using it here perhaps more in terms of truth functions rather than propositional functions. This is how Boole concludes this sections:

> The purport of the last investigation will be more apparent to the mathematician than to the logician. As from any mathematical equation an infinite number of others may be deduced, it seemed to be necessary to shew that when the original equation expresses a logical Proposition, every member of the derived series, even when obtained by expansion under a functional sign, admits of exact and consistent interpretation. (ibid., 69)

Perhaps Boole did not develop the notion of functions to represent predicate and relational propositions, but what he has suggested here is potentially much more powerful a tool that can be used for logic than what in fact Frege actually accomplished with function and argument.

In the final section ON THE SOLUTION OF THE ELECTIVE EQUATIONS Boole remarks that solutions of elective equations are similar to those of linear differential equations (ibid., 70).

Finally, at the end of the POSTCRIPT, Boole makes the theme of his work quite apparent: 'The general doctrine of elective symbols and all the more characteristic applications are quite independent of any quantitative origin.' (ibid., 82) This is where we began, based on the development of symbolical algebra that divests itself of quantitative origin to construct an algebra of logic that does not have any quantitative origin.

4.3.5 ***The incompletion of the emergence of modern logic***

Let us first see which characteristics of modern logic emerged in this work. With the use of algebra, formalisation of logic took a quantum leap. Proofs became simpler and at the same time more rigorous with the algebraisation of logic as demonstrated by Boole's algebrisation of classical logic. There is a definite transformation from the classical logic categorical propositions of classes to the elementary compositions and compounds thereof on which modern logic constructs a propositional calculus. A definite turn is made in concretizing an axiomatic system for symbolic logic. In sum we may say that in displaying all these features of modern logic whatever remained potential from Aristotle to Leibniz (almost bursting into actuality with Leibniz) was actualized by Boole and that is why he can be claimed to be the founder of modern logic. However, even though the mathematisation of logic had definitely begin with Boole, a lot of what he started here was quite incomplete. Neither a propositional calculus nor an axiomatic system of the algebra of logic are properly developed here. Quantification of predicates which Hamilton and De Morgan had introduced and Boole was well aware of it are hardly discussed and neither is the logic of relations. Furthermore, other features of what we have characterized as modern logic in section 1, such as development of modal logic, proofs of consistence and completeness have not even been attempted in this work.

What we have found in the unfolding of the narrative of the *Mathematical Analysis of Logic* is that throughout the work Boole is quite aware of the importance of logic both historically as well as at the present and in the future. Hence, Boole's interest is not simply to create an algebra of logic as an exercise of mathematics and to display a shining example of symbolical algebra by applying it to logic, but it is also and perhaps more so to develop logic itself, which has lagged behind precisely because mathematicians have not taken proper interest in it.

5. Conclusion

Perhaps the most important factor that stands out in the development of modern logic that I have not discussed in detail in this presentation is the axiomatisation of arithmetic. As we all know, geometry was axiomatised from the time of Euclid in 3rd century BCE. But arithmetic as we also know did not get axiomatised until the Nineteenth century by Dedekind and Peano. The fact that logic could not be properly axiomatised until arithmetic was axiomatised shows the intimate connection between arithmetic and logic that did not exist between geometry and logic. This has led to the view of logicism among many mathematicians and philosophers. However, it would be a mistake to claim that this is the only alternative. Another important factor was the formalisation of axiomatic systems and the metamathematics of proving consistency and completeness. For this we should hail Hilbert more than he has been given credit for in the development of modern logic.

If I were to pick out the major philosophers and mathematicians responsible for the emergence of modern logic they would be as follows in chronological order: Thales, Pythagoras (both for introducing deductive proofs I mathematics), Aristotle for inventing formal logic, Euclid for providing close to a complete axiomatised geometry, Leibniz for efforts in constructing a universal logical calculus, Peacock and Gregory for the development of symbolical algebra, DeMorgan for De Mogran's laws and quantifiers, Boole for the algebra of logic, Dedekind and Peano for the axiomatisation of arithmetic, Peirce and Schröder for the development of the logic or relations, Frege for his notations, concept of number, the use of function and argument and quantifiers, Hilbert for his metamathematics and finally at this 100th anniversary of the publication of *Principia Mathematica*, Russell and Whitehead for binging it all together.

There is a curious paradox of the intimate connection between arithmetic and logic. If axiomatisation is one of the necessary components of modern logic then modern logic definitely did not emerge until after the axiomatisation of arithmetic in the nineteenth century. Yet, the emergence of modern mathematics happened in what is acknowledged as the golden age of mathematics perhaps beginning with Descartes, Fermat and Pascal in the seventeenth century and reaching its peak with Euler, Lagrange and Laplace in the eighteenth century and ending with Gauss and Galois at the beginning of the nineteenth century. In the this development axiomatisation of arithmetic had not yet occurred but the most profound and long lasting contributions to algebra and analysis in the early nineteenth century had already been made. Why was there not a parallel golden age of logic during this two hundred period? It thing the reasons are complex but it does show that the connection between logic and arithmetic may not be as intimate as it is taken to be.

One answer that has been discussed in my presentation is that logic had not yet been algebrised. If Leibniz's attempt at algebrisation of logic had been successful as Descartes' algebrisation of geometry, then the parallel golden age of logic would also have flourished. Hence, we have a double edged assessment of Leibniz. Despite Leibniz's invaluable contribution to modern logic as Couterat has acknowledged, Leibniz was also responsible for the stalling of the progress. We could blame Leibniz for not working full time on this project and wasting too much time with theology, metaphysics and other matters. But the defenders of Leibniz would argue that if Leibniz did not operate as an encyclopaedic dilettante that he would not even have come up with the idea and attempt of developing a universal logical calculus to begin with. Furthermore, why place all the burden on Leibniz? Why could his successors like Wolff and Kant not have carried forth the task?

The proper answer is that other factors were missing as well, that some development of symbolisation in algebra itself was necessary, as we have clearly seen in this presentation, for the algebrisation of logic, which must precede its axiomatisation. Even if Leibniz had been successful in algebrising logic and building a sentential calculus, modern logic still would not have emerged because a clear notion and notations of quantifiers was not available until Frege for the development of a predicate calculus; and because the metamathematics of the Hilbert programme of proving consistency and completeness of any formal system and hence of the propositional and predicate calculus combined together as first order logic was not available in any complete sense of 'formal logic'.

For those who may be feeling that I have given an overdose of Leibniz let me end with a quotation from a mathematician whose contribution to logic has not properly been discussed in this presentation and this quotation succinctly captures the theme that I began with in the Introduction and ended with at the end of section 4:

> We know that mathematicians care no more for logic than logicians for mathematics. The two eyes of exact science are mathematics and logic, the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it sees better with one eye than with two. [De Morgan, 1858, 71] (Valencia 2004, 392)[8]

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[8] I have gotten this quotation and the reference of DeMorgan in the references from Valencia's article.

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January 14, 2009