The Completion of the Emergence of Modern Logic from Boole's *The Mathematical Analysis of Logic* to Frege's *Begriffsschrift*

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Abstract. Modern logic begins with Boole's *The Mathematical Analysis of Logic* when the algebra of logic was developed so that classical logic syllogisms were proven as algebraic equations and the turn from the logic of classes to propositional logic was suggested. The emergence was incomplete as Boole algebraised classical logic. Frege in *Begriffsschrift* replaced Aristotelian subject–predicate propositions by function and argument and displaced syllogisms with an axiomatic propositional calculus using conditionals, *modus ponens* and the law of substitution. Further Frege provided the breakthrough to lay down the groundwork for the development of quantified logic as well as the logic of relations. He achieved all of this through his innovative formal notations which have remained underrated. Frege hence completed the emergence of modern logic. Both Boole and Frege mathematised logic, but Frege's goal was to logicise mathematics. However the emergence of modern logic in Frege should be detached from his logicism.

Keywords: Boole, Frege, conditional, *modus ponens*, propositional calculus, quantifiers, function and argument, axioms.

1 Introduction

Can we pinpoint the moment when modern logic emerged? I propose that the emergence occurs in 1847 in Boole's *The Mathematical Analysis of Logic* [1] and in 1879 in Frege's *Begriffsschrift* [2, 3]. However this is to be complemented by the metalogic developed by Hilbert. Boole mathematised logic; whereas Frege raised logic to its highest pedestal by attempting to logicise mathematics and for Hilbert both logic and mathematics as a unified enterprise reached a second order level never developed before.

'Modern logic' may have various meanings such as (a) a fully developed first order propositional and predicate calculus, or (b) the development of higher order logics and metalogic including the relation of semantics to proof theory.[1] If 'modern' means latest, then (b) would be more appropriate. However, if 'modern' is taken in an

[1] I thank an anonymous referee for making this distinction clear and asking me to make clear in which sense I meant 'modern logic'.

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historical sense, then (a) may be the more important meaning. 'Modern philosophy' is taken to begin with Descartes in the 17th century. In this historical sense 'Modern' is a break from medieval philosophy. 'Modern philosophy' includes the span of philosophers from Descartes to the 21st century. No crucial break from classical logic occurred in the 17th or 18th centuries, hence we mark the 19th century as when modern logic emerged. Just as we say that Modern philosophy begins with Descartes, I want to claim that modern logic begins with Boole and Frege. Hence, under 'modern logic' I include not only (a) and (b) but also the more recent developments such as modal logic, epistemic logic, deontic logic, quantum logic, multivalued logic, fuzzy logic and paraconsistent logic. However, I use 'emergence' as being restricted to (a). Just as the emergence of Modern philosophy in Descartes does not include Kant's critical philosophy, similarly, the emergence of modern logic does not include (b) and later developments. Nonetheless just as Descartes is a necessary antecedent for Kant's critical philosophy, so (a) is a necessary antecedent for (b). Second order and higher order logics would not have emerged without the initial emergence of first order axiomatic propositional and predicate logic.

When I pinpoint Boole (1815–1864) and Frege (1848–1925) as the founders of modern logic, by no means do I wish to marginalise the contributions made to the emergence of modern logic by great mathematicians and logicians such as DeMorgan (1806–1871), Lewis Carroll (1832–1898), MacColl (1837–1909), Peirce (1839–1914), Schröder (1841–1902), Peano (1858–1932) and others. Each of these may have made as great or greater contributions than Boole and Frege and their influence on Frege at least may also be of great significance as he may not have been able to develop the logic he did without them. Rather, I have pinpointed one work of each and I wish to consider these works in themselves, in as much as possible, independent of their authors, as this makes it very convenient to understand when and how modern logic emerged. Hence, even though Frege on the whole may not be sympathetic with Boole's algebraisation of logic, I have attempted to show how the *Begriffsschrift* complements *The Mathematical Analysis of Logic* in the emergence of modern logic. I am open to alternative or complementary accounts which may for example consider DeMorgan and Peirce as the founders of modern logic. I also make a heuristic distinction between the roots of modern logic, the emergence of modern logic and the development of modern logic. The roots may go as far back as Aristotle, but the emergence occurs in the 19th century and most of what happens in modern logic in the 20th century may fall under the development of modern logic. My concern in this paper is only with the emergence of modern logic. As I have dealt extensively with Boole's *The Mathematical Analysis of Logic* elsewhere [4], I will concentrate in this paper on Frege's *Begriffsschrift*.

The emergence of modern mathematics came in the golden age of mathematics from mid 18th century to early 19th century led by Euler (1707–1783), Lagrange (1736–1813), Laplace (1749–1827) and Gauss (1777–1855). Non-Euclidean geometry and abstract algebra also emerged in this period. The delay in the emergence of modern logic was because of certain historical developments in algebra. Non-Euclidean geometries had already raised the possibility of a geometry that did not deal solely with measurement. With the development of symbolical algebra it became possible to have a purely abstract algebra that did not deal with quantity.

Hence, Boole could develop the algebra of logic which was completely symbolic and completely devoid of content. However, despite laying the groundwork of algebraic logic as well as a propositional calculus, Boole stuck to the Aristotelian limitation of subject–predicate propositions. So his logic was only partially modern.

Frege put an end to subject–predicate propositions and syllogisms by considering propositions in terms of functions and arguments. So, he also mathematised logic, but his mathematisation was conceptual and this logic would serve all of mathematics. Frege hence displaced the portion of logic that had remained Aristotelian and modern logic emerged. Even though '*Begriffsschrift*' translates as 'conceptual content', it was his innovative notations that finalised the revolution in logic and a comprehensive propositional and predicate symbolic logic could then be developed in 1910 by Whitehead and Russell in *Principia Mathematica* (*PM*) [5].

2 The Beginning of Modern Logic in Boole's *The Mathematical Analysis of Logic*

Symbolic propositional and predicate calculus could not be developed because neither Aristotle nor any logician after Aristotle was able to mathematise logic. Leibniz anticipated the algebra of logic to be the art of combinations as Louis Couturat states:

> Leibniz had conceived the idea […] of all the operations of logic, […] was acquainted with the fundamental relations of the two copulas […] found the correct algebraic translation of the four classical propositions, […] discovered the principal laws of the logical calculus, […] he possessed almost all the principles of the logic of Boole and Schröder, and on certain points he was more advanced than Boole himself. (my translation) [6, pp. 385–6]

What Leibniz really needed was the development of symbolical algebra which occurred more than a century after his death. In 1830 George Peacock claimed that operations in symbolic algebra must be open to interpretations other than that in arithmetic:

> […] in framing the definitions of algebraical operations, […] we must necessarily omit every condition which is in any way connected with their specific value or representation: […] the definitions of some operations must regard the laws of their combination *only* […] in order that such operations may possess an invariable meaning and character, […] [7, pp. viii–x]

The primacy of combinations over what they combine is thereby established.

Boole developed a quantity free algebra of logic in *Mathematical Analysis of Logic*.

He began by laying down the foundations of the algebra of logic which is a logic of classes in which the three combinatory laws (1) $x(y + z) = xy + xz$ (distributive), (2) $xy = yx$ (commutative) and (3) $x^2 = x$ (index) (p. 15); when combined with the axiom that equivalent operations performed on equivalent subjects, produce equivalent results, constitute the axiomatic foundations for all of logic [1, p. 18]. First, Boole represents

Aristotelian categorical propositions as algebraic equations. Then, he captures valid syllogisms of classical logic by multiplying equations and eliminating y which represents the traditional middle term:

$$ay + b = 0$$
$$a'y + b' = 0$$

When y is eliminated this reduces to:

$$ab' - a'b = 0 \qquad [1, p. 32]$$

Boole then makes the crucial turn to propositional logic in his account of conditionals. First he presents conditionals in terms of classes as in syllogistic logic:

If A is B, then C is D,
But A is B, therefore, C is D.

But then he expresses it in terms of propositions without reference to classes:

If X is true, then Y is true,
But X is true, therefore, Y is true.

> […] Thus, what we have to consider is not objects and classes of objects, but the truths of Propositions, namely, of those elementary Propositions which are embodied in the terms of our hypothetical premises [1, pp. 48–9].

We can embalm page 48 as the long awaited turning point from classical to modern logic as a scheme to translate syllogisms into inferences involving conditionals is suggested and in the particular example, the rule of inference of *modus ponens* is given in its propositional conditional form as we know it today.

Using 0 for false and 1 for true Boole now comes up with the possibilities for truth tables [1, pp. 50–51] and goes on to define conjunction, disjunction (both exclusive and inclusive), and conditional truth functionally [1, pp. 52–4]. As truth values are algebraised, mathematics can provide important insights into logic. These equations can be used for understanding truth functionality in a way that may not be understood without mathematics. The equation for the exclusive disjunction 'Either x is true or y is true' is $x - 2xy + y = 1$, which is acquired from the second and third row of the truth table: $x(1 - y) + y(1 - x) = x - xy + y - xy = x - 2xy + y$, and this must be true, so $x - 2xy + y = 1$. Now, since $x^2 = x$, we get: $x^2 - 2xy + y^2 = 1$. Which reduces to $(x - y)^2 = 1$; $x - y = \pm 1$. When x is true having the value of 1, then y must be false having the value 0 and when x is false, having the value 0, then y is true, having the value 1 to satisfy the equation [1, p. 55]. Hence, we see from the inside of Boolean algebra how a simple algebraic operation, but without regard to quantity, as the rule $x^2 = x$ is not a rule of ordinary algebra, leads to a clear definition of a logical operation like exclusive disjunction.

Boole concludes: 'The general doctrine of elective symbols and all the more characteristic applications are quite independent of any quantitative origin' [1, p. 82]. Boole successfully developed the algebra of logic on the basis of symbolical algebra that divests itself of quantitative origin. However, Aristotelian logic was sustained as is clear by the title 'Aristotelian Logic and its Modern Extensions, Examined by the

Method of this Treatise' of the culminating chapter of the logic part of his major work *Laws of Thought* in 1854. [8, pp. 174–86]. Clearly then, modern logic had not yet emerged in 1847 or in 1854.

3 The Completion of the Emergence of Modern Logic in Frege's *Begriffsschrift*

Frege completed the emergence of modern logic: first, by his innovative notation for judgments where the content stroke represented the content of the judgment, he finally brought down the axe on subject–predicate propositions which Boole was unable to do; second, he introduced a formal notation for conditional statements which in turn led to the development of axiomatic logic as well as a rigorous proof technique using *modus ponens* that made Aristotelian syllogisms archaic; third, he introduced a perspicuous notation for the universal quantifier so that a predicate as well as propositional calculus could be developed; fourth, he imported from mathematics the notions of function and argument and placed them at the core of symbolic logic and there was no looking back. Frege made up for Leibniz's failure to develop modern logic due to a lack of formalisation of relations and modern logic finally emerged. By no means was Frege a lone ranger in the emergence and development of modern logic. Invaluable contributions, without which Frege would have been nowhere, were made by DeMorgan, Schröder, Peirce and others. Yet Frege perhaps put it all together better than anyone else. The master historians of logic, Kneale and Kneale, best capture Frege's contribution:

> Frege's *Begriffsschrift* is the first really comprehensive system of formal logic. Aristotle was interested chiefly in certain common varieties of general propositions. He did indeed formulate the principles of non-contradiction and excluded middle, which belong to a part of logic more fundamental than his theory of the syllogism; but he failed to recognize the need for a systematic account of primary logic. Such an account was supplied, at least in part, by Chrysippus; but neither he nor the medieval logicians who wrote about *consequentiae* succeeded in showing clearly the relation between primary and general logic. Leibniz and Boole, recognizing a parallelism between primary logic and certain propositions of general logic about attributes or classes, worked out in abstract fashion a calculus that seemed to cover both; but neither of these enlarged the traditional conception of logic to include the theory of relations. Working on some suggestions of De Morgan, Peirce explored this new field, and shortly after the publication of the *Begriffsschrift* he even produced independently a doctrine of functions with a notation adequate for expressing all the principles formulated by Frege; but he never reduced his thoughts to a system or set out a number of basic principles like those given in the last section. Frege's work, on the other hand, contains all the essentials of modern logic, and it is not unfair either to his predecessors or to his successors to say that 1879 is the most important date in the history of the subject. [9, pp. 510–11]

Others have also expressed highest praises for the *Begriffsschrift*. von Heijenoort says 'Modern logic began in 1879, the year in which Gottlob Frege (1848–1925) published his *Begriffsschrift*' [10, p. 242]. According to Haaparanta 'Still others argue that the beginning of modern logic was 1879, when Frege's *Begriffsschrift* appeared' [11, p. 5]. Christian Thiel states 'If Frege has been regarded as the founder of modern mathematical logic, this characterization refers to his creation of classical quantificational logic in his *Begriffsschrift* of 1879 without any predecessor' [12, p. 197].

I now proceed to capture Frege's contributions in the *Begriffsschrift*.

The Preface announces Frege's motivation as he believes that pure logic gives the most reliable proof, and this depends solely on those laws on which all knowledge rests. Aristotle felt that his greatest achievement in logic was the discovery of the laws of thought. Boole appropriately entitled his later book as *An Investigation into the Laws of Thought*. There is a remarkable structural affinity among Aristotle, Boole and Frege, yet they are the greatest revolutionaries in logic. Frege, as a philosopher, made explicit what Boole as a mathematician left only as implicit. Boole algebraised logic by importing symbolical algebra into logic but at the same time, he set up formal logic that could become the basis of algebra as well. In attempting to logicise arithmetic, that is, to make it bereft of facts, and thereby content, Frege wanted to express arithmetical sequences by representing the ordering of a sequence without bringing in intuition and the existing mathematical language made this a very difficult task. Hence, he created his own formula language, the central nerve of which is *conceptual content* (*begrifflichen inhalt*) [2, pp. iii–iv; 3, pp. 5–6][2]. This formula language is modelled after the formula language of arithmetic, yet it is the 'formula language for pure thought' including arithmetic. Frege next announces that argument and function replaces subject and predicate of traditional logic and this will stand the test of time [2, p. vii; 3, p. 7]. And indeed it has stood the test of time. The Preface ends:

> As I remarked at the beginning, arithmetic was the point of departure for the train of thought that led me to my ideography. And that is why I intend to apply it first of all to that science, attempting to provide a more detailed analysis of the concepts of arithmetic and a deeper foundation for its theorems. [2, p. viii; 3, p. 8]

This is a profound insight that arithmetic is begging for someone to build logic out of its language so that it (the new logic) can provide a sounder foundation for arithmetic itself.

Part I is 'I. DEFINITION OF THE SYMBOLS'. In #1 Frege begins with the distinction between two types of signs; letters, like *a*, *b*, *c*, etc., that represent variability in meaning and symbols like +, -, √ , and 1, 2, 3, which have determinate meaning: '*I adopt this basic idea of distinguishing two types of signs*, which unfortunately is not strictly observed in the theory of magnitudes, *in order to apply it*

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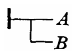
2 Though my reading is of the English translation of the *Begriffsschrift* [3] and all except one of the quotations are from the English translation, nonetheless I also give the citation of page number from the German original first [2] so that the reader can refer to the original as well. Hence, the citations are given as here '[2, pp. iii–iv; 3, pp. 5–6]'. If only the German is referred to than [3] is left out.

in the more comprehensive domain of pure thought in general' [2, p. 1; 3, pp. 10–11]. Frege steals an important distinction from under the noses of mathematicians, which the mathematicians do not clearly see, and builds on it the new logic. #2 introduces '-—' for expressing judgments. The horizontal stroke is the content stroke representing the thought of the proposition and the vertical stroke is the judgment stroke [2, pp. 1–2; 3, pp. 11–12]. #3 begins: 'Eine Untersheidung von *Subject* und *Prüdicat* findet bei meiner Darstellung eines Urthiels *nicht statt*' [2, pp. 2–3]. {'The distinction between *subject* and *predicate* does *not occur* in my way of representing a judgment' [3, p. 12]}. This marks the death of Aristotelian logic and the emergence of modern logic. '-—A' does not represent a subject–predicate proposition such as (1) 'Archimedes perished at the capture of Syracuse' but it represents the *conceptual content* of it, which is equally captured by a distinct subject–predicate proposition such as (2) 'The capture of Syracuse led to the death of Archimedes' or (3) 'The violent death of Archimedes at the capture of Syracuse is a fact'. As in (3) all judgments may be thought of as having a propositional content as the subject and 'is a fact' as the common predicate that makes them true [2, pp. 2–3; 3, pp. 12–13].

In #4 many distinctions of classical logic are dissolved such as that between universal and particular judgments which is now the distinction between universal and particular content and not of categorically different propositions. Negation is an adjunct to the content so that negative judgments are not categorically different from positive ones. Boole represented the four Aristotelian categorical propositions as algebraic equations hence dissolving the categorical distinction between particular and universal on the one hand and negative and affirmative on the other, since as algebraic equations they are not categorically distinct. Frege builds on this accomplishment; since, from this point on, there is no need to deal with Aristotelian categorical propositions [2, pp. 4–5; 3, p. 13]. #5 introduces the notation for conditional judgments as:

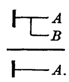


We symbolise this today as $B \supset A$. The horizontal lines to the left of *A* and *B* up to the middle vertical line are the content strokes of *A* and *B* respectively, and the horizontal stroke to the left of the top node of the vertical line is the content stroke of the meaning of the conditional regardless of the contents of *A* and *B*. First Frege gives four possibilities:

> (1) *A* is affirmed and *B* is affirmed; (2) *A* is affirmed and *B* is denied; (3) *A* is denied and *B* is affirmed; (4) *A* is denied and *B* is denied. [2, p. 5; 3, p. 13]

Then he defines the conditional as 'the judgment that *the third of these possibilities does not take place, but one of the three others does*' [2, p. 5; 3, p. 14]. This truth functional definition is the key that explains the axioms as laws of thought. Hugh MacColl had 'argued that the basic relation in logic is not class inclusion but implication between two propositions' [13, p. 373]. #6 develops *modus ponens* from the conditional as the only rule of inference:

(XX)::



Where (XX) represents |—B and the '::' indicates that B would have to be formulated and put into the inference [2, p. 8; 3, p.16]. Boole's algebraised logic required disjunction and conjunction as the main connectives which are symbolised by addition and multiplication respectively. Boole also gave the example of *modus ponens* when he made the transition from Aristotelian syllogisms to modern propositional logic on page 48. However, Boole failed to develop or even set the ground for an axiomatic propositional calculus which is the task that Frege picks up. Frege truth functionally explains *modus ponens*; |—B⊃A, |—B, therefore |—A as 'Of the four cases enumerated above, the third is excluded by |—B⊃A and the second and fourth by |—B so that only the first remains' [2, pp. 8–9; 3, pp. 15–16], that is, that both A and B are true. Hence, the truth of A is affirmed by this inference. The more general rule of *modus ponens* is given as:

> For, the truth contained in some other kind of inference can be stated in one judgment, of the form : if *M* holds and if *N* holds, then Λ holds also, or, in signs,
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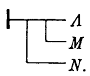
> 
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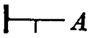
> From this judgment, together with |—N and |—M, there follow, as above, |—Λ. In this way an inference in accordance with any mode of inference can be reduced to our case. Since it is therefore possible to manage with a single mode of inference, it is a commandment of perspicuity to do so. [2, p. 9; 3, p. 17]

Since Frege, in the axiomatic development of logic, it has become almost a commandment to use *modus ponens* as the only rule of inference along with the rule of substitution. Frege makes conditionals foundational to the development of propositional logic with his notation of the conditional which is also used here to represent the rule of inference of *modus ponens*.

#7 introduces negation notationally as:



The small vertical stroke in the middle is the negation stroke and this represents 'not *A*' which means that the content of *A* does not take place. To the left of the negation stroke the horizontal line is the content stroke of the negation of *A* regardless of what *A* is, whereas the horizontal line to the right of the negation stroke is the content stroke of *A* [2, pp. 10–11; 3, pp. 17–18]. This seems to be a cumbersome way to represent negation. Why could Frege not simply have used '–A' or '~A' as we use today? The significance here has to do more with the philosophy of logic than with logic proper. With Frege's notation we can read negation as either 'not A is asserted' or 'the assertion of A is denied'. In the representation '~A' the literal reading when A is 'The Statue of Liberty is in Delhi' is 'it is not the case that the Statue of Liberty is in Delhi', since the negation sign is outside the sentence. However, with Frege's notation both 'The Statue of Liberty is not in Delhi' and 'it is not the case that the Statue of Liberty is in Delhi' are being expressed literally. If we read the negation as first the denial of A (as the content) and then the assertion of the denial; then the literal reading is that of 'The Statue of Liberty is not in Delhi'. However, if we read the negation stroke as the denial of the affirmation of A, then the literal reading is 'It is not the case that the Statue of Liberty is in Delhi'. Frege most probably would go for the first reading which is also the ordinary language reading. Nonetheless, the possible ambiguity here is a virtue as we can have both the ordinary language reading and the formal reading of the negation stroke simultaneously.

In #8 Frege defines '$\vdash (A \equiv B)$' as 'the sign *A* and sign *B* have the same conceptual content' [2, p. 15; 3, p. 21].

#9 introduces the notions of function and argument:

> *If in an expression, whose content need not be capable of becoming a judgment, a simple or a compound sign has one or more occurrences and if we regard the sign as replaceable in all or some of these occurrences by something else (but everywhere by the same thing), then we call the part that remains invariant in the expression a function, and the replaceable part the argument of the function.* [2, p. 16; 3, p. 22]

Whereas 'the number 20' is an independent idea, 'every positive integer' is an idea that depends on the context. Functions and arguments may be determinate or indeterminate, or more determinate and less determinate. Also, two different statements may be thought of as having the same function and two different arguments thought of as the same argument with two different functions. The definition of function and argument is not sufficient to claim that it can be implemented formally in logic. Hence, the symbolic representation in #10 is necessary to ground function and argument as foundational for modern logic: $\vdash \Psi(A, B)$, i.e., '*A* stands in the relation Ψ to *B*' [2, p. 18; 3, p. 23]. The logic of relations can now be developed. Whereas Boole algebraised logical propositions he could not make the distinction between propositional logic and the logic of relations. Frege finally makes this distinction as a statement in the form of function and argument is about conceptual content or formal content and not about subjects and predicates.

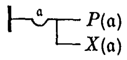
#11 '*Generality*' introduces the universal quantifier:

$$\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-\, \Phi(\mathfrak{a})$$

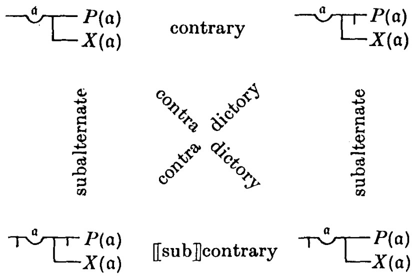
The horizontal stroke to the left of the concavity is the content stroke of the circumstances that whatever we put in for the argument $\Phi(a)$ holds; and the horizontal stroke to the right of the concavity is the content stroke of $\Phi(a)$ where we must imagine something definite for the argument *a* [2, p. 19; 3, p. 24]. Frege provides the notational representation of a bound universal quantifier just 12 lines below his formal representation of function and argument. '…the use of quantifiers to bound variables was one of the greatest intellectual inventions of the nineteenth century' [9, p. 511]. This use was developed from DeMorgan and Schröder to Peirce.

Frege's share of the emergence of modern logic begins on page 2 with the introduction of the notation that expresses the conceptual content and assertion in the judgment stroke and the declaration on pages 2 and 3 that his formalised representation of judgments no longer requires subject–predicate propositions; on page 16 with the definition of function and argument combined with the formal representation of functions within the judgment stroke on page 18; and on page 19 with the notational introduction of the universal quantifier. Complemented by Boole these pages of *Begriffsschrift* more or less complete the emergence of modern logic.

In #12 Frege uses his fundamental notation to express universal propositions:



i.e., 'if something has the property *X*, then it also has the property *P*' [2, p. 23; 3, p. 27]. Though this looks very different from the accepted notation today of '(x)(Fx⊃Gx)' we can clearly see how the '*a*' in the dip binds the conditional so that the notion of bound variable is concretised.[3] Is this the A statement of Aristotle? Boole had already suggested the transition from the propositions of terms to conditional propositions. Frege with his notation of conditional and the universal quantifier could hence represent A statements in their proper conditional form. This is how modern logic subsumes classical logic while eliminating subject–predicate propositions and syllogisms. And all of this can be done because of the notational representation of Frege here. Boole algebraised A propositions as: $x(1-y) = 0$, which means that either $x = 0$, or $y = 1$; that is, in 'all humans are mortal' either the antecedent '*x* is a human' is false or the consequence '*x* is mortal' is true. Frege replaces A propositions with: 'if something, *a*, has the property of being human, then it also has the property of being mortal', or '*a* is not mortal' is denied. Frege is aware that in the Aristotelian A propositions the subject term must designate something existent so that 'all unicorns are one-horned' is false because unicorns don't exist. In Fregean logic it is true as it is expressed in the above form, replacing *P* with *Ω* and *X* with *O*, where *Ω* stands for 'one-horned' and *O* stands for 'unicorns'. It is true because the case of *O*(*a*) holding, that is, being a unicorn, and *Ω*(*a*) not holding, that is, not being one-horned, does not occur precisely because there are no unicorns. What happens in the Boolean representation? Either '*x* is a unicorn' is false or '*x* is one-horned' is true. Since 'x is a unicorn' is never true, the A statement is always true. So Boole's representation of A statements denies the Aristotelian prerequisite of existence, therefore it is not an A statement. Therefore Frege is correct in replacing classical A statements rather than representing them. Frege uses the Aristotelian methodological model of the square of opposition



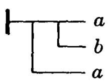
[2, p. 24; 3, p. 28]

[3] I thank another anonymous referee who suggested that I incorporate a conceptual analysis of why and how Frege's notations work. I have attempted a beginning of such a conceptual analysis in my comments on the conditional, negation, the universal quantifier and on *modus ponens* below. More detailed analyses of these will have to be undertaken in future in another paper.

but refuses to use the Aristotelian labels of A, E, I and O, because of their metaphysical commitment to existence. Hence we have here the desired emergence of modern predicate logic that transcends Aristotelian logic.

Part II is entitled ‘REPRESENTATION AND DERIVATION OF SOME JUDGMENTS OF PURE THOUGHT’. Frege provides the axioms of first order propositional and predicate calculus and proves numerous theorems. In #13 Frege, like Aristotle, Leibniz and Boole, wants to find the minimum laws of thought on which all of logic can be built [2, pp. 25–26; 3, 28–29]. The method that follows pronounces this minimalist program. Most logic books today begin with a list of axioms and then, without questioning whether they are the correct axioms, derive theorems from these axioms. This is the classic model of Euclid’s *Elements* [14], beginning with 23 definitions, 5 postulates (axioms) and 5 common notions and the propositions (theorems) of geometry are then one by one proved from these foundations. In successive theorems previously proved theorems are used as justifications for steps in the proofs. Frege preserves this structure of the Euclidean proving procedure. However, he does not provide all the axioms at the beginning. Rather the axioms are spread out throughout part II, each being stated when its need occurs. The axioms themselves emerge, so that we can pause and critically examine each axiom and convince ourselves that it really is a law of thought. Euclidean geometry could also have followed this Fregean method, as for example, the notorious postulate 5 is not used until proposition 29 [14, pp. 311–312]. It may have been better to state the postulate when it was needed after proposition 28. If Euclid had followed the Fregean method, then he himself would have wondered whether this postulate was correct when he stated it, as he may not have found it to be a law of thought but rather conjectural. Non-Euclidean geometry then would have begun with Euclid himself as the possibility of alternative postulates to postulate 5 or no postulate in place of it would have been entertained. Frege avoids this deficiency of the Euclidean system as in his system the grounds for alternative axiomatic systems to his own are provided since in the progressive emergence of axioms, any axiom may be replaced by alternative ones that then would lead to alternative theorems. If there are alternative axioms to any of Frege’s axioms, then these would have to be tested to see whether they really are laws of thought. Frege scholars seem to have overlooked this Fregean insight on method which is not to be passed over as an idiosyncrasy.

#14 provides the first axiom {A1}[4]:



[2, p. 26; 3, p. 29]

$q \supset (p \supset q)$[5]. This is not an axiom of *PM*, but is prominent as it is listed as the first theorem:

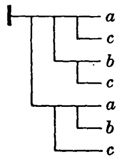
> The most important propositions proved in the present number are the following :
>
> *2.02. $\vdash : q . \supset . p \supset q$.

[4] All the labels for axioms, {A1} thorough {A9} are mine.

[5] Following each representation of an axiom in Fregean notation I immediately provide a symbolization of it in terms of notations that we are most familiar with.

> *I.e.* q implies that p implies q, *i.e.* a true proposition is implied by any proposition. [5, p. 103]

The reason makes it sound like a law of thought, then why did they not consider it as an axiom? Many logicians since *PM* nonetheless have followed Frege in making his first axiom, the first of their sets of axioms: Church [15, p. 72], Imai and Iséki [16, p. 19], Mendelson [17, p. 35]. Frege next gives the second axiom {A2}:



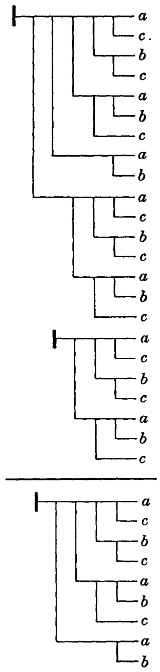
[2, p. 26; 3, p. 29]

$[p\supset(q\supset r)]\supset[(p\supset q)\supset(p\supset r)]$. This is second axiom for Church [15, p.72] and Mendelson [17, p. 35].

In #15 Frege proves the first theorem from {A1} and {A2}:

1

2



(3). [2, p. 30; 3, p. 32]

(3) says 'If b implies a, then any proposition c is such that "c implies that b implies a" implies that if "c implies b" then "c implies a"'. This may not be a law of thought, yet it is an immediate consequence of {A1} and {A2} which are laws of thought. Frege's proof employs *modus ponens* and the rule of substitution. From the Fregean picture we can clearly see that 1 is obtained by substitutions in {A1}, the substitution table being given on the previous page [2, 29; 3, p. 31]. 2 is simply {A2}. Since 2 is the bottom part of 1, that is the antecedent, hence we can deduce the top part, that is,

the consequent, which is formula (3), the first theorem. We can formulate this derivation as follows:

1. $\{[p\supset(q\supset r)]\supset[(p\supset q)\supset(p\supset r)]\}\supset\{(q\supset r)\supset([p\supset(q\supset r)]\supset[(p\supset q)\supset(p\supset r)])\}$

<substituting in {A1} ‘$q\supset r$’ for ‘p’; and ‘$[p\supset(q\supset r)]\supset[(p\supset q)\supset(p\supset r)]$’ for ‘q’>.

2. $[p\supset(q\supset r)]\supset[(p\supset q)\supset(p\supset r)]$ <A2>

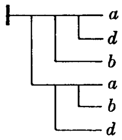
Therefore, 3. $(q\supset r)\supset([p\supset(q\supset r)]\supset[(p\supset q)\supset(p\supset r)])$ <1, 2, *modus ponens*>.

Frege’s proof takes up much more space but it may actually be easier to follow as he provides a table for substitution as well. The diagram proof may be more easily understood by those whose right hemisphere is relatively more dominant, whereas the relatively more left hemisphered activity persons would see the three line proof as much more perspicuous and convenient and Frege’s proof as cumbersome, difficult to type and wasteful. It is surprising that great logicians and mathematicians actually saw this as a fault in Frege. In 1918 C. I. Lewis stated:

> Besides the precision of notation and analysis, Frege’s work is important as being the first in which the nature of rigorous demonstration is sufficiently understood. His proofs proceed almost exclusively by substitution for variables of values of those variables, and the substitution of defined equivalents. Frege’s notation, it must be admitted is against him: it is almost diagrammatic, occupying unnecessary space and carrying the eye here and there in a way which militates against easy understanding. It is probably this forbidding character of his medium, combined with the unprecedented demands upon the reader’s logical subtlety, which accounts for the neglect which his writings so long suffered. But for this, the revival of logistic proper might have taken place ten years earlier, and dated from Frege’s *Grundlagen* rather than Peano’s *Formulaire*. [18, p. 115]

After high praise of Frege for providing proofs that proceed from axioms and the rule of substitution alone, bringing in modern logic proving techniques as replacement of syllogistic proofs; Lewis then criticises Frege for his space occupying notations. Why does his notation and proof procedure ‘militate against easy understanding’? Lewis does not explain. In 1914, Jourdain remarked: ‘[…] the using of FREGE’S symbolism as a calculus would be rather like using a three-legged stand-camera for what is called “snap-shot” photography [19, p. viii].’ Why are snap-shots to be preferred? Rather, Frege’s three-legged stand-camera photography is profound. The diagram of the full proof of (3) is an aesthetic marvel as it beautifully depicts how Frege moves naturally from conditionality to the first two axioms as laws of thought to the use of rule of substitution and *modus ponens* to derive the first theorem of his propositional calculus in formula (3). *Modus ponens* is appropriately often called the ‘conditional elimination rule’ (‘$\supset$ elimination’ or ‘$\rightarrow$ elimination’). When we look at the picture, we see the chunk of 2 as the bottom half of the chunk of 1 and eliminate it to end up with the top chunk of 1 as the conclusion, that is, formula (3). Modern logicians may not find this clarity of seeing very appealing and instead demand a 'metamathematical demonstration'[6]. Frege, being a philosopher, is closely connected to ordinary language and thought. Since *modus ponens* is not stated as an axiom but a rule of inference, we may ask whether it too is a law of thought. Frege might respond that it may not be an obvious law of thought as the axioms are, but it is a rule of inference that no one would deny, but everyone would accept, and that is why the visual mechanism used by him seems very convincing. The Fregean notationalised proof is the perfect complement to Boole's algebraisation of logic as there is a parallel here to Boole's representation of inferences as elimination in simultaneous equations. The two procedures are quite distinct, but there is a structural similarity in that the conclusion is being reached from the premises by using some type of elimination. That is why I use 'parallel' instead of 'similar'. Pages 29–30, where the proof for formula (3) is diagrammed should be added to the stock of pages where modern logic emerges. With axiomatic proof procedure the emergence is now approximately complete since Boole's algebra of logic was not an axiomatic system as both Peirce and Schröder realised [20, p. 59].

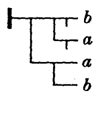
#16 provides the next axiom {A3}:



[2, p. 35; 3, 36]

$[p\supset(r\supset s)]\supset[\mathrm{r}\supset(p\supset s)]$. Is this a clear law of thought? Frege says that the antecedent, the bottom conditional in his picture, amounts to 'the case in which *a* is denied and *b* and *d* are affirmed does not take place', and the antecedent conditional 'means the same'. It means the same because of the commutativity of conjunction, that '*b* and *d*' is the same as '*d* and *b*', which is a law of thought. We know that '$p\supset(r\supset s)$' is the same as '$(p\&r)\supset s$' and Frege also knows that but he has not introduced conjunction as a logical connective. Hence, his reading in terms of a conjunction is a purely truth functional reading from the truth table. Why does Frege pick 'd' instead of 'c' here? Surely it makes no difference which letter one picks, though normally one would pick 'c' which is in sequence after 'a' and 'b'. My sense is that Frege wants to make this stand out as an axiom, and as the third axiom. The first axiom contained only 'a' and 'b', the second axiom contained 'a', 'b' and 'c', so let the third axiom contain 'a', 'b' and 'd' to make it stand out in comparison to the second axiom. In the rest of the section where formulas (9) through (27) are proven (without any more axioms appearing) only letters in sequence appear, which seems to confirm that Frege uses 'd' here to make the axiom stand out as an axiom. In *PM* {A3} appears as proposition *2.04. $\vdash:.p.\supset.q\supset r:\supset:q.\supset.p\supset r$ [5, p. 103]. However, this is an axiom neither for Church nor for Mendelson.

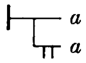
#17 provides the next axiom, {A4}:



[2, p. 43; 3, p. 44]

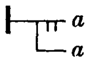
[6] This point was again raised by the first anonymous referee.

(p⊃q)⊃(~q⊃~p). In *PM* this is *2.16. ⊢:*p⊃q*.⊃.*~q⊃~p* [5, p. 107]. In #18 we get {A5}:



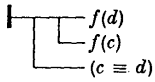
[2, p. 44; 3, p. 45]

~~p⊃p. In *PM* this is *2.14. ⊢.~(~*p*)⊃*p* [5, p. 106]. In #19 we get {A6}:



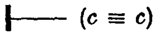
[2, p. 47; 3, p. 47]

p⊃~~p. In *PM* this is *2.12. ⊢.*p*⊃~(~*p*) [5, p. 105]. In #20 we get {A7}:



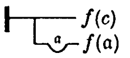
[2, 50; 3, p. 50]

(c≡d)⊃[*f*(*c*)⊃*f*(*d*)], i.e., whenever the content of *c* and *d* are the same then in any function we can substitute *d* for *c*. If all propositions can be expressed in terms of function and argument, then this also serves as the law of substitution which has been used all along. #21 provides {A8}:



[2, 50; 3, 50]

This is obviously a law of thought as it is the Aristotelian law of identity, except here it is in terms of the self identity of the content of a proposition. #22 provides the final axiom {A9}:



[2, p. 51; 3, p. 51]

(*x*)*f*(*x*)⊃*f*(*c*), which is universal instantiation. Is it a law of thought? Everyone would agree that if all humans are mortal then Socrates is mortal if he is a human. Whereas Boole demonstrated how the algebraisation of Aristotelian logic made a distinction between two types of invalid syllogisms that Aristotelian logic did not make [1, p. 41], Frege shows the reverse, that a distinction made between two types of valid syllogisms in Aristotelian logic, namely Felapton and Fesapo is not made here since their logical form is the same despite different placements of the subject and predicate in the first or major premise [2, pp. 51–2; 3, p. 52].

We come to the end of our trek of the emergence of modern logic in the *Begriffsschrift*.

Part III begins with a definition and derives theorems by applying the axioms to the definition, so that a theory of sequences is put forth which uses only axioms of logic that are pure laws of thought, the definition is also purely logical, and no intuition is involved which may be involved in the mathematicians' account of sequences. This is the beginning of the logicist program, that of reducing all of mathematics to logic. Most Frege scholars find a necessary link between Frege's logicism and his development of logic. Peter Sullivan states:

> Frege's major publications represent three stages in the project that occupied the core of his working life. Subsequently termed 'logicism', [...] In *Begriffsschrift* (1879) Frege set out the system of logic without

> which rigorous demonstration of the logicist thesis could not be so much as attempted, and illustrated the power of his system by establishing general results about sequences, including a generalization of the principle of mathematical induction. [21, p. 660]

Peirce argued that though logic depends on mathematics, mathematics does not depend on logic [22, p. 96]. This is not inconsistent with Frege's logicism for both Boole and he have shown the dependence of logic on mathematics as modern logic would not have emerged without the emergence of symbolical algebra and without the mathematical notion of function and argument. Modern logic once it emerges can be used to recapture mathematical theories solely by using laws of thought and logical inferences. Mathematics however does not depend on this logic and mathematicians may develop new theories without framing them in terms of modern logic. The logicist's task then is to track these theories down and reduce them to pure logic. Peirce also made a subtle distinction between mathematics as the science which draws necessary conclusions and logic as the science of drawing necessary conclusions [22, p. 95]. To the mathematician it does not matter how the necessary conclusions are drawn so intuition could well play a role, whereas for a logician conclusions must be drawn by using laws of thought and purely logical inferences. Frege's contribution to the emergence of logic is therefore not necessarily tied to his logicism even though his motivation for developing it is logicism. I differ from most Frege scholars in that I believe that whatever Frege's motivation may have been, the first two parts of the *Begriffsschrift* are autonomous and self-sufficient and bring about the completion of the emergence of modern logic that began with Boole. I stress this since in the development of logic since Frege mathematical logic is often linked with logicism whereas this need not be the case.

One meaning of 'modern logic' as discussed earlier includes the development of higher order logics. Part III does undertake this to some extent as Jose Ferreiros states: 'Upon more careful reading it becomes clear that Frege's system is higher-order throughout, and that he actually deployed higher order tools (this is explicit in the theory of series in the last part of *Begriffsschrift*)' [23, p. 444]. Using my heuristic distinction mentioned at the beginning of the paper, I would not go as far as to say that higher order logics emerged with Frege, but rather the roots of higher order logics are found in Frege, with the emergence and development of higher order logics to come later.

4 Conclusion

Mathematics and philosophy are the proper parents of logic as both are second order disciplines dealing with pure form without content and unlike the sciences they are not directly about the world. Adamson states: 'The distinction of logic from the sciences, as dealing in the abstract with that which is concretely exemplified in each of them, […]' [24, p. 9]. Hence, the biggest names in the origins, emergence and development of formal logic and the roots and emergence of modern symbolic logic are of philosophers like Aristotle, Leibniz, Peirce, Frege, Russell and C. I. Lewis; and mathematicians like Boole, DeMorgan, Schröder and Hilbert. I have pinpointed the emergence of modern logic from Boole's *Mathematical Analysis of Logic* to Frege's

Begriffsschrift. The emergence of modern logic begins with Booole's book where logic was algebraised, and particularly on page 48 where the transition from the Aristotelian logic of classes to propositional calculus is strongly suggested. In Frege's book we find the near completion of the emergence with his innovative notation of the judgment and content strokes that allows us to replace subject–predicate propositions of classical logic [2, pp. 1–3]; of providing a notational representation of the conditional along with a truth functional definition to be read out with the picture of the notation [2, p. 5]; of replacing subject–predicate propositions by function and argument [2, p. 16, 18]; of the symbolic representation of the universal quantifier (p. 19); and the technique of proving theorems by using axioms, the sole inference rule of *modus ponens*, and the rule of substitution [2, p. 30]. The completion of the emergence of modern logic came with the development of metamathematics and metalogic in which the soundness, consistency and completeness of axiomatic propositional as well as predicate logic can be proven.

I have argued that the emergence of modern logic in Boole is partial as he ends up algebraising Aristotelian syllogisms. Frege lays the grounds for the completion of the emergence as the classical pillar of subject–predicate propositions is replaced by functions and arguments, and that of syllogisms is replaced by axioms and axiomatic proofs. By making Aristotelian logic archaic and replacing it with symbolic axiomatic logic Frege completed the task that Boole began and for all intents and purposes from 1847 to 1879 modern logic finally emerged. I say 'finally' because whereas modern science and modern philosophy are usually marked as having emerged in the 17th century and modern mathematics in the 18th century, modern logic did not emerge until the 19th century. Whereas Boole algebraised logic borrowing from mathematics, specifically from symbolical algebra, the idea of combinations without regard to content; Frege borrowed from mathematics the notions of function and argument, but his real intention was to logicise arithmetic once modern logic itself had emerged. Some would argue that the completion did not take place until the beginning of the 20th century until Russell and Whitehead's *Principia Mathematica* and Hilbert's metalogic. I would label these as 'the development of modern logic' rather than the 'emergence of modern logic'. However, I will not push this distinction here within the scope of this paper and will concede to those who want to include these latter contributions as part and parcel of the emergence of modern logic.

Frege's contribution can best be assessed by highlighting an important distinction as stated by Jourdain:

> […] the distinction pointed out by LEIBNIZ between a *calculus ratiocinator* and a […] *lingua characteristica*. […] The objects of a complete logical symbolism are: firstly, […] providing an *ideography*, in which the signs represent ideas and the relations between them *directly* […], and secondly, […], from given premises, we can, in this ideography, draw all the logical conclusions which they imply by means of rules of transformation of formulas analogous to those of algebra,—in fact, in which we can replace reasoning by the almost mechanical process of calculation. This second requirement is the requirement of a *calculus ratiocinator*. It is essential that the ideography should be complete […] {and} concise. The merits of such an ideography are obvious: rigor of reasoning is ensured by the calculus character; we are sure of not

> introducing unintentionally any premise; and we can see exactly on what propositions any demonstration depends. We can […] characterize the dual development of the theory of symbolic logic during the last sixty years as follows: The *calculus ratiocinator* aspect of symbolic logic was developed by BOOLE, DE MORGAN, JEVONS, VENN, C. S. PEIRCE, SCHRODER, Mrs. LADD FRANKLIN and others; the *lingua characteristica* aspect was developed by FREGE, PEANO and RUSSELL. […] FREGE has remarked that his own symbolism is meant to be a *calculus ratiocinator* as well as a *lingua characteristica* […] [19, pp. vii–viii).

Lingua characteristica provides an ideography in which the signs represent concepts and the relations among concepts and *calculus ratiocinator* is the rigorous proofs from axioms in this ideography. Frege, according to the findings of this paper, is right on the mark in claiming that his symbolism is both a *calculus ratiocinator* and a *lingua characteristica* because not only do his notations capture the relations among concepts but they themselves are conceptual inventions, and the understanding of the axioms as laws of thought as well as the proving techniques are totally dependent on the notations and formulations using these notations themselves, hence Frege also provides a *calculus ratiocinator*. The emergence of modern logic is incomplete in Boole who provides a rigorous *calculus ratiocinator* by using algebra to capture logic, but fails to create a new ideography to relate concepts, but more or less accepts the old ideography of Aristotle which is not a universal *lingua characteristica*. Frege creates this new ideography and also provides a rigorous *calculus ratiocinator* for his new *lingua characteristica* which is more universal than Aristotle's, hence satisfying Leibniz's dream, as it can be used not only for all of mathematics, but for science and philosophy as well. I began the discussion of Frege with how he stole an important distinction from under the noses of mathematicians. I end here with a distinction that I steal from the mathematician Jourdain to elevate Frege to the rank of the greatest logician, whereas Frege was mostly ignored by mathematicians for about three decades after the publication of the *Begriffsschrift*.

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