PHILOSOPHY AND PHENOMENOLOGICAL RESEARCH
A Quarterly Journal

VOLUME XXI, No. 2 DECEMBER 1960

"7 + 5 = 12" AS A SYNTHETIC PROPOSITION *

I propose to discuss once again Kant's claim that (most) arithmetical propositions are synthetic. I want to bring out its truth together with the importance of his view that numbers are relations of objects to their parts (in contrast with a fashionable view that numbers are at least classes of classes of objects).

My interest, more than in the historical side, lies in the development of insights which Kant did not formulate with completeness.

In the first part I discuss rather briefly Kant's own idea of a synthetic proposition, and in Part II I formulate and apply a test for deciding whether a proposition is synthetic or not. My claim is that such a test embodies and develops the substance of Kant's conception of synthetic propositions. In Part III I develop what I take to be Kant's views of ordinary arithmetical propositions, and argue in some detail for their plausibility.

I. SYNTHETIC PROPOSITIONS

1. *Characterization:* It is well known that by synthetic propositions (or judgments) Kant meant propositions which are neither analytic nor self-contradictory (K. d.r. V., A, 150–6). If we leave aside compound propositions (e.g., disjunctions, conditionals) explicitly recognized by Kant (*ibid.*, 73 f.; B, 140 f.) every (simple) proposition is for him made up of a subject and a predicate. Since "modality of judgments ... concerns only the value of the copula in relation to thought in general" (A, 74), we can limit ourselves to categorical propositions, which are all of the subject-predicate form. Now, if an analytic proposition "is affirmative, I only predicate of a concept what is already contained in it; if it is negative, I only exclude from it its opposite" (A, 154). Thus, Kant means by a synthetic categorical proposition (or judgment) one in which the predicate asserts something neither contained in, nor opposite to, the subject.

* I am very grateful to Mr. Richard Robinson (Oxford University) for his kindness in reading this paper. Thanks to him the style has greatly improved and many obscurities have been removed. Prof. H. Paton has assured me that I have not misinterpreted Kant.

Practically everybody fails to notice Kant's discussion of negative judgments, but it is frequently argued [1] that 'contained in' is a vague, metaphorical expression, which rather obscures Kant's definition of 'analytic' and 'synthetic.' Doubtless there is plenty of obscurity in the expression and, hence, in the definition. Kant never clarified the connection between analyticity and definitions (*ibid.* 728 ff.; *Prolegomena,* 2c). However, there are abundant clear-cut cases to make exaggeration unnecessary. For example, with regard to a given perfect classificatory system, a genus is contained in its species; likewise, the several species (or determinates) under the same genus (or determinable) are opposite to each other. Thus, Kant would certainly have said that the following are analytic:

(1) Everything red is colored.

(2) No patch is both green and red all over.

The terms (or concepts) 'red,' 'colored,' and 'green' are certainly empirical, not only in the sense that they are used to identify (or think about) sensible qualities, but also in the sense that they cannot be defined by means of any formula like " 'bachelor' means an unmarried adult human male"; to learn the complete meaning of those words (or form the concepts) we must experience instances.

However, this is irrelevant to the analytic character of (1) and (2). It shows only that there is no specific difference that can be "added" to colored(ness) to produce red(ness), while the 'bachelor' formula collects several specific differences. But this does not establish that (1), e.g., is not analytic; it only establishes that *after* we have secured its analyticity by recognizing that being colored is the genus of being red we cannot satisfy a *further* demand – to exhibit a specific difference. Even though the color words are indefinable in that sense, to learn the meanings of 'blue,' 'red,' 'colored,' etc., involves learning that red, blue, etc., *are* precisely colors, i.e., involves learning their classificatory connections.

2. *Subject-Predicate Form:* Nowadays it is customary to sneer at Kant's characterization of analytic and synthetic propositions in terms of subjects and predicates. The charges are:

(a) compound propositions do not have a subject-predicate form;

(b) existential propositions have no predicate, since, as Kant emphasized (A, 597 ff) existence is not a predicate;

(c) relational propositions have several subjects (and no predicate).

Charge (a) has been most recently formulated by Pap (*op. cit.* 27 *f.*). He considers the form "all *a* are *b*, but some *a* are not -*b*" and asks: "where is 'object,' where 'predicate'?" (*ibid.*, 28). Now, this being a com-

[1] Cf. Arthur Pap, *Semantics and Necessary Truth* (New Haven: Yale Univ. Press. 1958), p. 30; Richard Robinson, "Necessary Propositions," *Mind.* N.S. LXVII (1958), p. 297.

pound statement form, nothing prevented Kant from saying that in Pap's schema there are two subjects: the occurrences of '*a*,' and two predicates: '*b*' and 'not -*b*.' Since, by the ordinary rules of syllogistics, it entails "some *b* are not -*b*," which is a contradictory categorical proposition form, for the predicate danies what the subject asserts (A, 153). Pap's schema is certainly self-contradictory. It is true that Kant failed to discuss the formal logic of compound propositions; although he was not discussing formal logic at all, he could have written a paragraph on the analyticity of compound propositions.

(b) has been put forward by Pap (*op. cit.* 27) and mentioned by Robinson (*op. cit.*, 296 f.). However, the argument is based on hearsay. For in his discussion of the ontological argument Kant very explicitly: i) says that existence is not a *real* predicate; ii) asserts that "everything can serve as a *logical predicate*" ("Zum *logischen Prädikate* kann alles dienen, was man will, sogar das Subject kann von sich selbst prädiziert werden; denn Logik abstrahiert von allem Inhalte." A, 598: Kant's own italics); iii) explains the role of 'is' or 'exists' when these words function as logical predicates; to place the object(s) thought in the subject in the whole context of experience, as present in perceptions or as linked to something perceived ("durch die Existenz wird der Gegenstand aber als in dem Kontext der gesamten Erfahrung enthalten ... gedacht ... unser Bewusstsein aller Existenz aber (es sei durch Wahrnehmung unmittelbar, oder durch Schlüsse, die etwas mit der Wahrnehmung verknüpfen)." A, 600 f.). Furthermore, Kant iv) claims that the mistake of the ontological argument is to regard 'exist(s)' as a real predicate when it is just a logical predicate (A, 598).

Thus, Kant incurs no contradiction in both taking for granted that a proposition (or judgment) always has a (logical) subject and a (logical) predicate and arguing that existence is not a (real) predicate.

Charge (c) is often combined with (a), as in Pap's "If somebody is somebody's teacher, then somebody is somebody's pupil" (*op. cit.*, 27). Since this is a compound proposition, Kant could have said that it has two subjects and two predicates. A typical example which might be used to substantiate (c) is "Peter gave Mary a book for John." Here it is claimed that there is no subject-predicate form, for there are four subjects and one predicate ('giving'). However, classical logicians and Kant could have said that the logical subject is Peter and that the logical predicate is 'gave Mary a book for John." Similarly, in "If anybody is a student he has a teacher" the subject is the class of those who are students.

Charge (c) should have seemed justified only at the beginning of the development of the logic of relations; for, after Wiener [2] and Kuratow-

[2] Norbert Wiener, "A Simplification of the Logic of Relations," *Proceedings of the Cambridge Philosophical Society*, XVII (1914), pp. 387–390.

ski[3] had showed that relations can be formulated as classes, it was no longer correct to say that the subject-predicate form was inadequate for relational propositions. It could have been replied that for these two logicians relations turn out to be classes of classes, whereas we ordinarily take relations as characteristics of objects; i.e., as of the same logical type as the latter's qualities. This is a mere technicality. However, when Quine [4] showed that modern logical theory can construe relations as coextensive with classes, even that technicality was removed. Consequently, the sympathizers of Kant do not have to say, e.g., "that Kant's Formal Logic and modern mathematical logic are trying to do different things," they can acknowledge with a free conscience that modern logic has, in fact, come to develop the procedures for testing analytic propositions. Nevertheless, "the criticism that Kant ought to have done [in the Kritik] what mathematical logic does is an unreasonable criticism." [5] Nor do they have to try to attenuate charge (c) by adducing that "it is surely significant that, in our ordinary lives, we should cling with so much obstinacy to the old linguistic conventions, and that most of our statements should conform to the subject-predicate pattern." [6]

3. *Subjects and Predicates.* It has been seldom noticed that Kant's notion of logical subjects and predicates does not require them to be frozen in a given proposition. In discussing the principle of contradiction he makes it clear (A, 153) that the subject and the predicate of a proposition can often be subjected to several different arrangements. For instance, "Peter is a childless father," "A childless father is Peter," "Childless Peter is a father," and "Peter, a father, is childless," are all analyses or formulations of the same proposition; but only in the last two does the predicate deny what is "now a part of the concept of the subject." Similarly, "Peter gave Mary a book for John" can be parsed as having the logical subject 'Mary' and the logical predicate 'was given a book by Peter for John,' etc.

Thus, we can conclude that, according to Kant, a categorical proposition is synthetic if and only if in none of the formulations either of itself or of its denial does the (logical) predicate deny (part of) what the (logical) subject asserts.

---

[3] Casimir Kuratowski, "Sur la notion de l'ordre dans la théorie des ensembles," *Fundamenta mathematicae,* II (1921), pp. 161–171.

[4] W. V. Quine, "On Ordered Pairs," *Journal of Symbolic Logic,* X (1945), p. 95 f; "On Relations as Coextensive with Classes," *ibid.,* XI (1946), p. 71 f.

[5] H. J. Paton, *Kant's Metaphysic of Experience* (London: Allen & Unwin, 1936), Vol. I, p. 211.

[6] H. W. Cassirer, *Kant's First Critique* (London: Allen & Unwin, 1954), p. 177.

II. COLORS

1. *Perception and Intuition.* For Kant a proposition (or judgment) is certainly synthetic if to establish its truth (or falsity) we need "something else" (A, i, 155) beyond understanding the meaning of the subject. Although this is very vague language, it is clear from Kant's discussion that intuition (in his sense of the word) is a sufficient criterion – though not necessary (A, 162, 733, 149, etc.) – for the syntheticity of a proposition. By 'intuition' Kant usually means perception; but he states quite explicitly that there is a pure or *a priori* intuition of spatial and temporal relations (A, 25, 77, 99). His main view is, of course, that empirical intuition is possible only through a pure intuition (A, 28, 165, etc.). Thus, a judgment is synthetic if one of the following operations is required to establish its truth (or falsity):

(a) perception or empirical intuition,
(b) pure intuition as a feature of a perception,
(c) isolated pure intuition.

A mature person may be able to have isolated pure intuitions. But it must be confessed that Kant never offered a very clear account of pure intuition. His claim that it guarantees the *a priori* and synthetic character of "7 + 5 = 12" cannot be assessed easily. However, here I do not propose to investigate the nature of pure intuition. I want to formulate a criterion for the syntheticity of a proposition, which, I believe, clarifies the "something else" of which Kant spoke. Even if my criterion is not derivable from Kant's own statements, I am sure that it is quite within the spirit of Kant's arguments and views. In order to reach a good vantage point, I shall discuss that criterion in connection with nonmathematical propositions at first.

2. *Combinations of Colors.* Clearly, propositions ascribing colors to given objects are synthetic, for they have to be verified or falsified by perception: test (a). On the other hand, (1) and (2) above do not require a perceptual test; learning the meanings of color-words involves perception, no doubt, but it also requires learning the classificatory connections among color-words – and these must be learned as the framework for color experiences, not as empirical findings.

Consider now:

(3) Orange is the color resulting from mixing red and yellow.

This is not a statement of the classificatory connections among color predicates. Nor is it a definition of 'orange.' Even if it is uttered in answering the question "What is orange?" its role in such an answer is to give, not a formula for the substitution of synonymous expressions, but *just* a directive for the production of instances. True, in a sense when we give the meaning of a word by means of a formula, say, "Bachelor = unmarried adult male," we are also furnishing the language student with a directive for finding instances of bachelors. But, in this case, if he has succeeded in following the instructions correctly throughout he cannot fail to produce instances. Whatever he finds to be human, male, adult, and unmarried, *must* be a bachelor. On the other hand, it is quite possible for the pupil instructed on the meaning of 'orange' by means of (3) to follow the instruction contained therein to the last detail, and yet fail to produce orange. Surely, in such a case we must give him an explanation: that the pigments do not mix, that the illumination is not normal, that the disease he is suffering from prevents him from seeing the normal color of things, etc. However, this is only a sufficient mark of the synthetic character of (3), even though it is a necessary mark of its being empirical – for the failure to produce instances of orange is due to a deficiency of the materials or to a deficiency of perception.

On the other hand, the following features constitute both sufficient and necessary conditions of the mere syntheticity of (3), independently of its empirical character, i.e., regardless of whether or not there are synthetic nonempirical propositions. If (3) is used to teach a person the meaning of 'orange,' the pupil must understand

i) that he *could* have discovered the truth of (3) by himself, had a different teaching device been employed – provided, of course, that other devices are possible, as in fact they are possible in the case of 'orange': "Orange is the color of these objects [and here the objects are exhibited or pointed to]" or "Orange is the color of the fifth book on the small table in the next room";

ii) that he *might* certainly forget that (3) is true without thereby "losing" or distorting his concept of orange (or of red, yellow, color, etc.), for the possession of such a concept involves several abilities which do not include the knowledge that (3) is true: e.g., ability to recognize orange regardless of how it is produced, to form images of orange objects, etc.;

iii) that (3) is *just* a directive for finding or producing instances of orange, in which the only analytic component is the classificatory statement "Orange is a (nother) color"; he must understand a) that (3) is not a rule of universal substitution *merely* to effect the elimination of a longer expression, b) that objects can be orange without having been produced by mixing red objects with yellow objects, c) that objects can be orange even if there were neither red nor yellow objects.

The language student who wants to know what orange is does not have to say to himself at any moment: "Now I see that (3) is only a directive for the production of instances," etc. His grasping i)–iii) is shown both by the way he relates this new concept to those already in his possession and by the special use he assigns to it. Indeed, grasping i)–iii) is a part of assigning to the concept of orange both its proper use and its right position in the network of relations among concepts, i.e., in Kant's terms both its transcendental and logical (A, 268) places. There is no such thing as learning the meaning of 'orange' by ostensive "definition," in the sense that one's learning that meaning is merely to associate the noise with an item in perception, which item offers itself to him in an absolute metaphysical nakedness – as if each concept could stand alone in a logical universe of its own.[7] For, as Kant put it, "perceptions [e.g., of colors] without concepts [of colors] are blind," i.e., are no perceptions (A, 51).

It should be noticed that the above test i)–iii) of syntheticity runs counter to the not too uncommon assumption

(G) that statements employed in teaching the meaning of a word are either part of the meaning of that word, or formulate part of such meaning, or are part of the definition of such a word, or are analytically related to the meaning of the word in question.[8]

This assumption is active in most behavioristic views, which make behavior a criterion or a logically necessary and sufficient condition for the existence of, say, strong pains or great joys. It is also operative in views which regard entities like electrons, molecules, etc., as just theoretical constructs. Our criterion i)–iii) tries to make clear how synthetic propositions (or statements) are used to introduce new concepts, i.e., concepts synthetically related to the older ones used in their introduction.

It should be noted that criterion i)–iii) for the syntheticity of a proposition holds independently of Kant's views on mathematics, space, etc.

III. ARITHMETICAL STATEMENTS

1. *Concrete Addition.* It seems to me that Kant's argument to prove that (4) $7 + 5 = 12$

is synthetic can be favorably compared with test i)–iii). We saw that (3) also satisfies a stronger condition: that a person who followed the instructions it contains may fail to produce an instance of orange. Clearly, this is not true of (4), and here we find the reason why it is not an empirical proposition.

Obviously, (4) can be used to teach a person the meaning of the word 'twelve.' If this were not so, it would be difficult to hold that the meanings of the words 'twelve,' 'five,' 'seven,' are logically connected so as to make (4) analytic; indeed, not even assumption (G) could help us to argue that

[7] One of the latest defences of (un = Kantian) theories of pure givenness is found in Pap, *op. cit.*, pp. 215 ff., 224 ff., 240 ff., 244 ff.

[8] For an extreme application of this assumption *cf.* Pap, *op. cit.*, pp. 249–259, where Pap talks about atomic and existential propositions analytic by ostensive definition, such as "This object is blue" and "Something is red."

(4) is analytic. But if a four year old child knows the numerals from 1 to 10, and asks "What is 12?" one can easily instruct him by saying that:

(5) 12 is the number of things you get by putting 7 things and 5 things together. Clearly, if (5) and (4) are not synonymous, it should at least be conceded that (4) can be employed in teaching the meaning of the symbol '12.' But what can the difference between (4) and (5) be? There is no need to emphasize that the signs '5,' '7,' '12,' '+,' and '=' are taken in their most ordinary meanings, e.g., in the sense in which they are used both in counting and in measuring. The difference that stands out between (4) and (5) is that you do not have to *put* 7 and 5 things *together* to add them up. That is, (5) would seem to be only a special case of (4). You could say more generally

(6) 12 is the number of things you obtain by considering together 7 things and 5 things.

Now, i) *could* a person know that ordinary meaning of 'twelve' together with the meanings of 'five' and 'seven' and come to discover by himself that (5) or (6) is true? This does not seem implausible. A small child may have learned that five is the number of fingers on each hand, which is also the number of years he has lived; he may have also learned that seven is the number of hearts and diamonds in the corresponding bridge cards; he may have learned that twelve is the number of pennies he is given weekly by his parents, who always arrange them on a desk in two triangles, each made up of six pennies. The child may have grasped that the designs can be changed in both size and shape. Suppose then that one day our child is amusing himself with all his pennies, and that suddenly it dawns on him that seven pennies and five pennies are twelve pennies. Happy with this discovery he tries it with marbles, and with other toys – and finds that it works.

ii) It might be adduced that the child of the tale does not actually have the concepts of seven, five, and twelve as we normally understand them. It could be said, e.g., that he understands by these numbers the property of certain collections that you can arrange their members in some geometrical designs. However, the property in question is not just a geometrical property like having a certain shape or curved sides, for the designs can be varied – and it *is* in fact an essential part of each number that certain designs are possible and certain others are impossible. A person has not really learned the meaning of 'three' if he does not (at least confusedly) know that he cannot arrange three objects in a square (Cf. A, 140). Furthermore, even though we have been conditioned by Frege, Russell, and Whitehead to think of numbers as properties or classes of collections, this is not a self-evident assumption. It is in fact the claim I want to dispute in this essay.

But could a person forget the truth of (6) or (5) without thereby "losing" his concept of twelve? It seems that in one sense he *could* – for to know the meaning of '12,' in the most ordinary sense of the word, involves other abilities which are not impaired by ignoring the truth of (6) or (5). To repeat, to know the meaning of '12' is to be able to recognize certain designs as belonging to the class of designs with 12 points or 12 lines or 12 circles or 12 shadings, etc.

iii) Likewise, Kant could have said that (5) or (6) is *just* one directive, out of many others, for finding or producing instances of twelve. Other directives equally good are, e.g., "the number of marbles you obtain by joining two collections of six each" (and here it is not implied that numbers are properties of collections or classes), or "the number of fingers you find by combining all your fingers with those of my left hand [where the speaker has only two left fingers]."

Doubtless, (4) is intended as a formula for (almost) universal substitution. Yet it is not intended as a mere abbreviation for a longer expression, but as a more compact formulation of the general procedure (6) for the production or discovery of dozens. Clearly, dozens may be produced by other combinations of objects, and dozens could exist even if there were no aggregates of seven or five objects: the universe could be such that, e.g., every female always had twins or quintuplets in her fourth pregnancy and after that became sterile.

Thus, additive propositions are synthetic. They pass the test i)–iii) with flying colors.

2. *Numbers as Properties of Objects.* To the above account of the addition formulas it may be objected (a) that we could not learn the meanings of all numerals by ostension. Indeed, some psychologists hold that most people can only identify at a glance, without counting, up to six objects. This cannot be dismissed as just an empirical fact about human minds, for even if we could all be trained to identify 1000 or 1000000 objects in whatever design they are presented, the important thing about the number sequence is its infinity. So, at some stage, however late, we shall have to resort to a formula like

(7) the successor of $n = n + 1$.

And here it may be alleged that (7) or some other formula given to generate the sequence of (remaining) numbers is a formal definition, or the analytic consequence of such a definition.

It could also be adduced (b) that statements like (5) and (6) are really different from purely arithmetical statements like (4) and (7). Some will even go to the length of offering a theory as to how a merely formal system, like pure arithmetic, gains application to reality through an interpretation of some of its terms by means of rules of correspondence, or coordinating definitions, or operational rules, relating such terms to items in experience. This is, they will say, applied arithmetic. To avoid a digression, I just want to indicate that such a view of applied arithmetic is utterly inadequate for the simple reason that those correspondence rules will relate an abstract symbol to *our* ordinary numerals; i.e., the "interpretation" of the formal arithmetic requires *either* that we have already an informal arithmetic employed in counting and measuring *or* that we be able to identify the number of objects in any collection whatever its design or arrangement. In the former case there is a *petitio* and in latter the impossibility emphasized by objection (a). Thus, we cannot defend the analytic character of arithmetic at the cost of making applied arithmetic synthetic and empirical.

Kant seems to recognize that purely formal or abstract arithmetical propositions like (4) and (7) are different from the concrete ones like(5) and (6). But he insists that there is no such large gap between formal or analytic and applied or synthetic arithmetic; he argues that pure ordinary (or classical) arithmetic rests from the very beginning on applied arithmetic (Mathematik der Erscheinungen. A, 165) – and not the other way around. Mathematical propositions "would have no meaning, if we were not always able to show their meaning as applied to phenomena (empirical objects)" (A, 239 f.). It is the purpose of his principles of the axioms of intuition and of the anticipations of perception to emphasize the phenomenal sense of numerals (A, 163 ff., 167 ff.). However, Kant is quite clear on the impossibility of introducing (learning or teaching) every concept of number by instances [as we do with colors]. He explicitly acknowledges (A, 142) that the very concept of number includes (7) as its law of generation.

Before discussing Kant's adumbrated reasons for the syntheticity of (7) we should note that it is not true that (7) is the law for generating numbers which we employ in teaching the meaning of all numerals. It is probably most normally employed in teaching the meanings of small numerals; but when a person asks, e.g., "What is a million?" the most common answer is

(7a) One million is one thousand times one thousand.

This, of course, does not prove the syntheticity of (7); it only shows that (7a) should be subjected to the above given test i)–iii) to decide whether it is synthetic or not. But the important question still remains: How is it that, whatever law generating all numbers we use, since we *must* have one, such a law is not an analytic proposition?

To me the clue to Kant's view here lies in a passage often dismissed by his commentators:

> Ob er ["7 + 5 = 12"] gleich synthetisch [*a priori*] ist, so ist er doch nur ein einzelner Satz. So fern hier bloss auf die Synthesis des Gleichartigen (der Einheiten) gesehen wird, so kann die Synthesis hier nur auf eine einzige Art geschehen, wiewohl der *Gebrauch* dieser Zahlen nachher allgemein ist. (A, 164; his italics.)

Not content with saying that (4) is a singular, though synthetic, *a priori*, proposition, Kant proceeds immediately to contrast it with geometrical propositions like "A triangle can be drawn with three lines, two of which are together greater than the third," which, he holds, is universal. He argues that the lines can be of different sizes and directions, so that the terms 'line' and 'triangle' denote here real predicates, i.e., classes of objects. This distinction appears puzzling.[9] Yet Kant emphasized that the distinction between universal and singular propositions "deserves a special place in a complete table of the aspects of thought in general" (A, 71). Therefore, his assertion that arithmetical propositions are singular most probably comes from a careful thought.

Now to say that a proposition is singular is to say that its subject clearly designates one and only one object; i.e., the logical subject is either a proper name or a definite description of an object. Thus, when Kant says that (4) is a singular proposition, he is saying that 'the sum of 5 and 7' is the description of a single object, and since it is a proposition of identity, '12' is either a definite description or a proper name. Since he quickly adds that the numbers later receive a general *use*, Kant seems to be saying that the expressions '5,' '7,' and '12' are actually what we now call undetermined individual constants. (This terminology and the precise distinction between variables and constants were developed after Kant, and it is understandable that he found it difficult to make his point clear.) Thus, Kant is not saying that numbers are perfect, formal particulars, which exist in sensibility, like the particulars usually attributed to Plato in his allegory of the divided line (*Rep.*, 509 ff.). He seems to be saying that to talk of *the* numbers seven and five is just a general, unspecified way of talking about one or another individual which has a certain property, as when a writer introduces the *name* 'Metaphysicus' to refer to any metaphysician or when a geometrician introduces the *name* 'ABC' for any triangle that might be chosen and has the properties he is interested in examining.

The individuals which '7,' '5,' and '12' are names of must certainly be those which have the properties of being seven, five, and twelve, respectively. These individuals are the concrete aggregates or complex individuals made up of several objects, which are its parts (Cf. A, 162 f.). The individual aggregate of, or individual compounded out of, the fingers in my right hand is five, or has the property of being five, *in relation to its parts*. To be sure, this individual aggregate has no proper name, and most individuals of this sort, or of the simple sort for that matter, remain unnamed. Nevertheless, we do have descriptions for them, and occasionally give them names, as, e.g., 'The Twin Cities' employed in the United

---

[9] E.g., H. J. Paton, *op. cit.*, Vol. II, p. 130; he goes on to say that "the difference between geometry and arithmetic on which Kant is here insisting seems to be little more than a difference in expression" (p. 131).

States to name the individual formed by Minneapolis and Saint Paul, Minnesota.

Now, it seems that the ordinary role of numbers in experience is (at least in some respects) better described by the view I am here attributing to Kant:

A) We do not say in ordinary life, as the customary Frege-Russell reduction of mathematics to logic requires, that the class of the fingers in my right hand is five or belongs to five, or has the property of being five. These statements have a different "feel" from the ordinary statement "The fingers of my right hand are five." The reference to the plural parts is necessary. The former statements "feel" incomplete, as if the class of fingers having members but no parts were itself too simple or lacked complexity enough to exemplify a numerical property.

B) The ordinary concepts of numbers are such that at least oneness is a property of single objects: it is my hand which is one, not the class of objects whose only member is my hand. In fact, it is *the fingers together* which are five.

C) As we normally measure objects, it is they which have the property of being five feet tall or seven pounds heavy, etc. The measurement gives a relation of the object to its (possible) parts. It is of the nature of ordinary numbers to be properties of the same logical type, whether they appear in counting or in measuring.

D) We do see in the most literal sense (some of) the numbers objects exemplify. We *see* that the fingers of a hand are five just as we see their size and color. True, we do not see the threeness of the Holy Family – but we do not see the color of the other side of the moon either, or the color of objects in darkness; yet we can touch their plurality and feel their number. On the other hand, it is not clear that one sees second-order properties. I do not see the property that red and blue have of being colors, or the second-order property that being-to-the-left has of being transitive. Even more, on the customary Frege-Russell view numbers are classes of classes. But classes are things of a sort which cannot be correctly said to be either visible or invisible; and classes of classes are just more so.

E) As pointed out above, it will not do to regard (4) as analytic and (8) as synthetic:

(8) 7 apples + 5 apples = 12 apples.

As we normally apply (natural) numbers to objects, (8) is just a special case of (4): both are analytic, or synthetic, and both are of the same logical type and character. The Frege-Russell view, however, must assign them a different logical character or cannot regard (8) as a special case of (4). It defines (7 + 5) as the class whose members are the unions of disjoint (classes) members of 7 and 5. Thus, in '7 + 5' there are no variables that can be replaced by 'apples' to make '7 apples + 5 apples' a *mere* specification of '7 + 5.' At least it will be necessary to invoke additionally that the union of two subsets of the same class is also of the same class.

On the view I am assigning to Kant it is possible to distinguish between the nominal use of numerals in abstract additions like (4) and their adjectival use in concrete additions like (8) without distorting the similarity between (4) and (8). The nominal numerals are unspecified names of individual aggregates, whereas the adjectival numerals stand for descriptions of partially specified individual aggregates. (4) presupposes that the aggregates are homogeneous (gleichartig), and (8) just makes explicit one kind of homogeneity to which (4) applies.

Now both types of statement should be distinguished from the more basic, predicative type of statement like:

"There are 12 persons in the room."

"The persons who came to see you were five."

"The number of persons waiting for you is five."

"Five persons are waiting for you."

These propositions ascribe a numerical property to certain objects. They are basic in the sense that counting presupposes them from the very beginning, for counting is nothing but the method for determining how many objects of a given kind there are, i.e., the method for finding out what numerical property is exemplified by the objects in question, taken as an aggregate.

In fact even Frege acknowledged that "ordinary language does assign number not to concept but to objects: we say 'the number of bales' just as we say 'the weight of the bales' " (*Grund.* 64).[10] And an examination of Frege's arguments shows that they only prove that numbers are not empirical properties of simple objects, not (i) not nonempirical properties (ii) of compound individuals. Point (ii) is clear, e.g., in his argument.

> The green color we ascribe to each single leaf, but not the number 1000. If we call all the leaves of a tree taken together its foliage, then the foliage too is green, but it is not 1000. To what then does the property 1000 really belong? (*Grund*, 28).

Point (i) appears in his more important arguments:

> ....an object to which I can ascribe different numbers with equal right is not the real subject of the number predicates... The number 1... or 100 or any other number, cannot be said to belong to the pile of playing cards in its own right, but at most to belong to it in view of the way in which we have chosen to regard it... What we choose to call a complete pack is obviously an arbitrary decision, in which the pile of playing cards has no saying (*Grund*, 29).

[10] *Die Grundlagen der Arithmatik* (Breslau: W. Koebner Verlag, 1884; New York: Philosophical Library, 1950, trans. into English by J. L. Austin), to be referred to as *Grund*. This brief note on Frege was added on the suggestion of Professor Romane Clark (Duke University).

> It does not make sense that what is by nature sensible should occur in what is nonsensible (*Grund*, 31).

Clearly, that a person is both to the right and to the left of different persons does not prove that to-the-left-of and to-the-right-of are not properties of objects. (Indeed, we owe to Frege a good deal of our understanding of the nature of relations.) Frege's argument shows that it is not an empirical property of objects their being objects or parts. We have to decide what is to count as an object or part, and the decision is almost completely free. But once we have made such a decision it is independent of us whether the objects are 15 or 1000 or 2000000. Here is a pile of playing cards and we can choose what to call a part or unit. If we decide that it is a pile of objects I shall call "doubles," namely two cards with the same numeral and of the same color, then we may find that pile has the property of having 26 parts. This dependence of the numerical properties for their exemplification on our decisions as to what a unit or part will be is certainly a sign that numerical properties are not mere empirical properties. But it can be argued, as Kant did, that for experience to be possible we must be conscious of objects, i.e., discrete objects, whose determination as such objects is not the result of a decision in the normal sence.

3. *Abstract addition.* There is another point intimately connected with (4) being a singular proposition. Kant contrasted mathematical definitions with definitions in philosophy and in everyday life (A, 727 ff.). He asserted that mathematical definitions are synthetic, where 'mathematical' is to be understood in the sense of classical mathematics. Clearly he knew of definitions by genus and difference, and knew that such definition must be analytic, in the sense (as we would now put it) that their consequences are all analytic propositions. Therefore, Kant holds that even if, following (7), we define '12' by a formula like

(9) $12 = 11 + 1$,

we would not be introducing a concept by giving what it contains. Now, this can be appreciated on my view if we remember that for Kant the words '12,' '11,' and '1' are undetermined individual constants. Obviously, if these words stand for unspecified proper names, (9) and (4) do not include a single word which stands for a property or *real* predicate (A, 598). No characteristic is necessarily included in a proper name, so the predicate asserts nothing that is also asserted in the subject.

Here again we find a reason for rejecting assumption (G), *viz.*, that every statement employed to introduce a concept gives an analytic relationship of such a concept. Along the lines of the present interpretation of Kant, (9) is a definition in exactly the same sense in which (10) is:

(10) *Caesar* is (=) the Roman General who conquered Gaul and crossed the Rubicon with his army.

(10) may very well be used to introduce the actual, historical name

'Caesar' to a person. But the person will utterly fail to understand the lesson if he does not understand that 'Caesar' is a proper name, i.e., that (10) is not a rule for the elimination of a longer expression in favor of a shorter one. He must understand that (10) is a synthetic proposition, since it contains just a directive or recipe for some sort of identification of the man Caesar. He must understand that, while the term 'Caesar' appears in the identity statement (10) and may appear in many, none of these statements can provide a synonym for it.

In general, identity statements which are used to introduce proper names in our (or in somebody else's) language may be termed "definitions" because of their introductory role; but they are not definitions in the sense of mere rules of substitution or in the sense of having only analytic consequences. If numerals are proper names, unspecified or not, then the customary formulas introducing them in accordance with a generating principle like (7) cannot be said to be analytic. Thus just as the example of orange (3) help us to understand the addition of concrete numbers, the case of proper names like (10) help us to understand abstract additions.

Since numbers are relations of individuals to their (possible, or actual) parts, the fundamental meaning of addition involves the idea of compounding individuals. On the assumptions that: (i) 'x' and 'y' range over homogeneous (gleichartig) individuals, i.e., whose ultimate parts are all of the same kind, e.g., apples or fingers, and ii) '*m*,' '*n*,' and '*h*' are numerals, addition is defined by

(11) $m + n = h$, if and only if: given that x is n, y is m, and x and y have no common parts, then the compound x–y is h.

To describe fully the use of numbers in ordinary additions, countings, and measurings, we must mention (12) and (13) below. On the same assumptions (i) and (ii)

(12) $x \doteq y$ (read: x and y are arithmetically identical), if and only if they have the same number n.

(13) Every x which has no proper parts is 1.

(11) – (13) make it clear that the "application and meaning" (Haltung und Sinn) of ordinary numbers lie – primarily in discrete physical objects like "fingers, the beads of the abacus, or in the strokes and points" (A, 240)

The other formal properties of numbers are certainly those which Peano described in his famous axioms. And it is quite the thing for mathematicians to concentrate on such axioms and disregard (11) – (13). This maneuver lightens the weight of their principles and allows them to fly to new discoveries. They create new concepts of types of numbers; but this can in no way erase the fact that ordinary, phenomenal numbers are characterized by (11) – (13).

If one assumes throughout a universe of discourse of homogeneous individuals which satisfy (12), one has to mention neither the homogeneity nor the arithmetical identity. Numerals can then be regarded as the names of representative objects, whose purely arithmetical properties are described by, say, Peano's axioms. Actually, since we often need two objects with, e.g., 5 parts, as in "$5 + 5 = 10$," we allow ourselves the use of several representatives of the classes of objects with given n (possible) parts. Thus, the nominal numerals are not just unspecified proper names like 'Metaphysicus,' which corresponds to an individual's name like 'Georg Wilhelm Friedrich Hegel,' or 'Caesar' in (10). Since we allow ourselves several 7's and 5's, the numerals are unspecified family or group names like 'Smith' and 'Ernest,' in contexts in which any Ernest or Ernests may do and we are not at all interested in the differences or relationships between one Ernest and another (as in Wilde's play). At any rate, each occurrence of a numeral is the occurrence of an unspecified proper name, and the preceding examples (4)–(10) are all singular propositions in which the predicate does not repeat what the subject asserts, and are, therefore, synthetic.

The central difference between ordinary pure arithmetic and ordinary concrete arithmetic lies in the abstraction of the former from the latter, wich characterizes numbers by, say, Peano's axioms together with (11)–(13), the properties of the operation of compounding individuals, and the properties of the part-whole relationship, e.g., "the compound x–y = the compound y–x," "x–x=x," "x is a part of y, if and only if there is an individual z such that x–z=y." Once Peano's axioms are taken in isolation, the only basic statements for addition that remain are

(14) (a) $m + 1$ = the successor of m.

(b) m + the successor of n = the successor of $m + n$.

But at this stage there is a great temptation, particularly if one holds assumption (G) discussed above, to regard (14) as *the* definition of '+.' Again, nothing bad necessarily ensues from our calling (14) a definition, and in so calling it we are correctly emphasizing the fact that in an axiomatic formulation of ordinary arithmetic as a system of pure mathematics (14) does introduce the concept of addition. But a great wrong is done if, under our formal education in logical theory, we regard every formula or statement described as a definition by the practitioner of a discipline as being a definition in the logician's sense, i.e., as just a rule of verbal substitution or as an analytic identity of the concepts in question.

Obviously, (14) by itself introduces a concept of addition which may be inadequate to establish the truth of (4), "$7 + 5 = 12$," or (5) or (6). Suppose, for example, that the sequence of natural numbers as characterized by Peano's axioms alone is (c) 1, 3, 5, 7, 9, 10, 8, 6, 4, 2, 11, 13, 15, 17, 19, 20, 18, 16 . . .

Clearly, (4) interpreted in accordance with (14) and (c) is false; $7 + 5$, which is the successor of $7 + 3$, is the successor of the successor of the successor of 7, which is 8. On the other hand, (4) interpreted in accordance with (11) is true, and so are (5) and (6), provided that $5 = 4 + 1, 4 = 3 + 1$, etc. And as argued above, these identities are, like (10), the equations of a proper name with one description among a large number which are possible: "4 is the number of the larger part resulting from breaking a compound of 7 when the other part has 3," etc. Similarly, a formula like (7) or (14) (b) does not merely introduce a shorthand; it identifies natural numbers in their positions in the normal sequence. That is, (7) establishes the synthetic identity of the operation denoted by the sign '+' in (11) with the operation denoted by the sign '+' in (14). Hence at least one of these two propositions must be synthetic.

4. *Logicism.* Just one word on the already classical "proof" given by the Frege-Russell logicistic school that arithmetical propositions are analytic. There is no need to repeat the well-known difficulties of the axioms of choice and reducibility. The latter was eliminated and the former might very well be regarded as a logical truth. It is well-known that the axiom of infinity does look like a synthetic proposition in most of its formulations. But there are ways of avoiding that.[11] Thus, the fundamental difficulty remains in the fact that the formalization of mathematics and logic presupposes a very rich meta-language, in which numerals are employed quite freely without being reduced to logic.

But I am more interested in discussing something related to Kant's phenomenal view of numbers, which makes nominal numerals unspecified proper names of compound individuals. It is of the essence, i.e., analytically true, of a proper name *not* to be synonymous with any description of the object of which it is a name, just as much as it is of its essence that it may always be equated with one or more definite descriptions of the object in question. This latter characteristic has given rise to efforts like Quine's [12] to get rid of names in favor of definite descriptions. But they overlook one specific epistemological role of proper names, which is more akin to that of demonstratives like 'this' or 'that,' whose role of seizing or picking up features of present experience can be performed by no definite description. Proper names are like generalized demonstratives: they point beyond the *hic et nunc* thanks to their hold on one or more different descriptions of the same object. They link the different, unshared experiences of different persons and allow them to identify such experiences as relating to the same objects.

In a world in which no two persons had different experiences of the same objects there would be no need for proper names. In a world in which

[11] Cf. for one famous way W. V. Quine, "New Foundations for Mathematical Logic," *American Mathematical Monthly*. Vol. 44 (1937), pp. 70–80.

[12] E.g., W. V. Quine, *Methods of Logic* (New York: Henry Holt Co., 1955), pp. 220–224.

no two people shared a single experience, there would be room neither for demonstratives nor for proper names. In a world in which every two persons had exactly the same experience, demonstratives and descriptions would do – but then how could we tell that there were two persons?

If Kant is correct in claiming that numerals are unspecified proper names of possible objects (and we have seen that there are some reasons for his claim), then they are synonymous with nothing even though they may require analytically the truth of propositions equating them to some other expression. For instance, it may be an analytic truth that numbers can be obtained by operations on other numbers – yet the actual formula relating a given number to a specific operation on other given numbers may be synthetic. Once again, then, a purely formal elimination of ordinary numerals is plausible. Indeed, such an apparent elimination may be a great discovery in that we may learn more about the formal properties of numbers. But it will be apparent only, for the epistemological function of numerals, which is not a formal characteristic, will be missing. Thus, the most successful "reduction" of arithmetic to logic can only offer us a purely formal counterpart, or image, or model (in the logician's and mathematician's use of the terms), of the (informal) arithmetic of ordinary, phenomenal numbers. The most successful "reduction" of that arithmetic to logic is very much like a perfect painting of a house: here we have exactly how the house looks, we can learn from it to appreciate more the beauty of the garden from the angle chosen by the painter, we may realize for the first time that the gutters should be replaced, that it would be esthetically better to grow pink roses near the peach tree, etc. Yet there is just one thing which makes the house "irreducible": – its inhabitability.

It might be said that modern mathematics is essentially abstract and has nothing to do with those properties of compound individuals or with the relation of an object to its parts. This is certainly and fortunately true. But it only shows that that fundamental, classical phenomenal arithmetic (Arithmetik der Erscheinungen) is no longer what many mathematicians study, even that their abstract arithmetics are like abstract paintings of our phenomenal numbers. They have constructed other disciplines on its basis, which have doubtless turned out to be more interesting. On the other hand, there is no reason why Kant could not have joined us in acknowledging that a large part of what modern mathematicians study is formal logic – and that, contrary to his (original) belief, formal logic did advance beyond Aristotle. For instance, the theory of relations is logic, and a mathematical discipline like group theory is an exciting part of that logic.

HECTOR NERI CASTAÑEDA.

WAYNE STATE UNIVERSITY.